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VIBRATION EVALUATION OF SANDWICH CONICAL SHELLS WITH FIBER-REINFORCED COMPOSITE FACINGS

By

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Byron L. Mayberry
John D. Ray

December 1968

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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This research effort was carried out under Contract DAAJ02-67-C-0111 by the University of Oklahoma Research Institute to develop mechanical characteristics of fiber glass reinforced plastics.

This report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound.

This report is published for the dissemination of information and the stimulation of further research.

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FORT EUSTIS, VIRGINIA

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SUMMARY

This report describes an experimental and analytical evaluation of the vibrational characteristics of a truncated conical shell of sandwich construction. The shell was composed of fiber glass-epoxy facings and aluminum honeycomb core. The nominal shell geometry had a 5-degree semi-vertex angle, a diameter of 58 inches at the large end, and a length of 72 inches. These dimensions were selected to be typical of an Army aircraft fuselage structure. The shell was suspended by soft cords, to closely approximate free-free boundary conditions, and was excited laterally by an electrodynamic shaker placed at various locations along the shell. The shell surface was instrumented with 600 metallic-foil strain gages. Accurate separation of vibrational modes having closely spaced frequencies was accomplished by means of the Kennedy-Pancu method of data reduction, using measured frequency, strain amplitude, and relative phase angle between response and excitation. The two lowest unsymmetric-mode frequencies, for various values of circumferential wave number, agreed quite closely with those predicted by the Rayleigh-Ritz inextensional analysis presented in the report. This analysis includes the orthotropic nature of the composite-material facings. Values of the damping logarithmic decrement are also presented.

FOREWORD

This report was prepared by the University of Oklahoma Research Institute (OURI) under Phase I of U.S. Army Aviation Materiel Laboratories (USAAVLABS) Contract DAAJ02-67-C-0111 for research conducted during the period from July 1, 1967 to July 31, 1968. Work under this contract has also included biaxial loading of fiber-reinforced-plastic laminates (Phase II), reported separately in Technical Report 68-86, titled Behavior of Fiber-Reinforced-Plastic Laminates Under Uniaxial, Biaxial, and Shear Loadings.

The research effort is a continuation of the research accomplished under three previous contracts and reported in the following USAAVLABS reports:

1. Technical Report 67-65, Fabrication and Full-Scale Structural Evaluation of Sandwich Shells of Revolution Composed of Fiber Glass-Reinforced Plastic Facings and Honeycomb Cores.

2. Technical Report 65-66, The Effect of Resin Content and Voids on the Strength of Fiberglass-Reinforced Plastics for Airframe Use.

3. Technical Report 65-60, Dynamic Elastic, Damping, and Fatigue Characteristics of Fiberglass-Reinforced Sandwich Structure.

4. Technical Report 65-15, Strength Properties and Relationships Associated with Various Types of Fiberglass-Reinforced-Facing Sandwich Structure.

5. Technical Report 64-37, Research in the Field of Fiberglass-Reinforced Sandwich Structure for Airframe Use.

The present report was written by Dr. Charles W. Bert, project director and Professor of Aerospace and Mechanical Engineering; Mr. Byron L. Mayberry, research engineer; and Dr. John D. Ray, project director.

Special acknowledgement is made to Mr. W.C. Crisman and Mr. Frederick Lehmann in connection with specimen fabrication.

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LIST OF SYMBOLS

A, B	constants of integration appearing in Equation (2)
b	material damping factor, lb/in.
c_{ij}, d_{ij}	parameters defined by Equations (7)
D_θ	shell flexural rigidity in circumferential direction; see Equation (8)
$D_{s\theta}$	shell twisting rigidity; see Equation (9)
E_s, E_θ	Young's moduli of facing material in meridional and circumferential direction, respectively, psi
F	exciting force, lb
$G_{s\theta}$	planar shear modulus of facings, psi
G.F.	gage factor, dimensionless
h	distance between centroids of the facings, in.
k	spring rate in single-degree-of-freedom system; see Appendix
m	mass in single-degree-of-freedom system (Appendix); or designation of general modes
n	circumferential wave number
P, Q, R	parameters defined by Equations (6)
R'	resistance, ohms
R_1, R_2	radii of small and large ends of shell, respectively, in.
s	distance along meridian of middle surface of conical shell, measured from apex, in.
s_1, s_2	values of s at small and large ends of shell, respectively
t	time, sec
t_f	thickness of one facing, in.
u	meridional displacement, in.
v, w	circumferential and normal displacements, respectively, in.

V	normalized circumferential displacement, given by Equation (10); or voltage, volts
W	Normalized normal displacement, given by Equation (11)
x	instantaneous displacement, in.
X	system response factor = X_o/F
X_o	system response (displacement in Appendix; strain in Section 6)
α	semi-vertex angle of cone, deg
γ	completeness parameter = R_1/R_2
Δ	an incremental change in the quantity that follows
δ	logarithmic decrement; see Appendix
$\epsilon_{b\theta}$	circumferential bending strain, dimensionless
$\bar{\epsilon}_{b\theta}$	normalized value of $\epsilon_{b\theta}$, defined by Equation (13), dimensionless
ϵ, ϵ_d	strain and dynamic strain, respectively, dimensionless
θ	circumferential angular coordinate, deg.
κ_θ	change in curvature in circumferential direction, in.$^{-1}$
λ	eigenvalue; see Equation (4)
$\nu_{s\theta}$, $\nu_{\theta s}$	Poisson's ratios of the facing material, dimensionless
ν	Poisson's ratio of isotropic material, dimensionless
ρ	material density, lb-sec^2/in^4
ϕ	phase angle, deg
ω	frequency, rad/sec
ω_1, ω_2	half-power frequency points, rad/sec
ω_e	natural frequency associated with purely extensional motion, rad/sec
ω_i	natural frequency associated with purely inextensional motion, rad/sec

1. INTRODUCTION

Under the sponsorship of the U.S. Army Aviation Materiel Laboratories (USAAVLABS), the University of Oklahoma Research Institute (OURI) has conducted a major research program for the past five years in the area of fiber-glass reinforced plastics (FRP) suitable for primary structures of U.S. Army aircraft. Tasks which have been completed include the evaluation of:

(1) Several sandwich fabrication techniques (Reference 1).

(2) Effect of primary fabrication variables on conventional strength properties of laminates (References 1, 2, 3).

(3) Several honeycomb-core sandwich configurations as small panels sized to fail in face rupture and various buckling modes (References 1, 4, 5).

(4) Fatigue properties of sandwich material as beams with FRP facings and honeycomb core (References 6 and 7).

(5) Dynamic stiffness and damping of sandwich material in the form of beams (References 6 and 8).

(6) Fabrication and static buckling testing of full-size sandwich shell structures in the form of cylindrically curved panels, complete cylinders, and truncated cones (References 9, 10, 11, and 12).

(7) Mechanical behavior of flat laminates under uniaxial tension, biaxial tension (1:1 and 1:2 principal-stress ratios), and pure shear (Reference 13).

The objectives of the present research were (1) to measure resonant frequencies and associated modal strain distributions and damping factors of a truncated conical sandwich shell suspended in the free-free condition and (2) to compare these quantities with theoretical predictions.

A number of experimental investigations have been carried out to evaluate theories of vibration of sandwich-type beams and plates (Reference 14). However, the authors do not know of any investigation on sandwich-type shell structures, which are widely used in aircraft structures. The shell configuration used in the present investigation was that of a truncated cone. The scale selected was sufficiently large to be typical of an Army aircraft fuselage and to achieve lowest resonant frequencies sufficiently low to enable accurate measurement of a large number of natural modes. Also, it is difficult to scale down a sandwich-type structure; i.e., if the core thickness is too thin, the sandwich effect is negligible.

In the interest of economy, the shell was fabricated with the same equipment and techniques previously developed by OURI for the shell buckling

experiments (Reference 9). The inside and outside facings consisted of two plies of 828-Z epoxy resin resin reinforced by Volan-A-finished E-glass in the form of 181-style fiber glass fibric. The core material was 5052 aluminum-alloy, hexagonal-cell nonperforated honeycomb.

To be quite certain that the boundary conditions achieved in the experiments matched those used in analysis, the boundary conditions selected for the experiments were free edges. Also, these conditions were the least expensive to achieve experimentally and were used in previous experiments on homogeneous shells at the Langley Research Center (References 15, 16, and 17) and at Southwest Research Institute (Reference 18).

Although the free-free condition is most easily achieved experimentally, it is perhaps the most difficult condition from an analytical viewpoint, particularly in the case of a sandwich shell, because of the difficulty in finding functions that satisfy the boundary conditions. This is attested to by the very limited number of analyses of homogeneous conical shells (or even homogeneous cylindrical shells) with these boundary conditions. It can be applied to truncated conical shells of either simple single-layer or sandwich construction with either isotropic or orthotropic (composite-material) facings. Inextensional motion, in which the extensional strains are neglected, is assumed. Solution is carried out by the Rayleigh-Ritz energy method for the two inextensional modes, which are the lowest unsymmetric modes. The circumferential wave number, n, may be any integer greater than one. The validity of the simplifying assumptions made in the analysis are verified by the good agreement between the experimental and analytical results.

2. SPECIMENS AND MATERIAL

Two large-scale sandwich conical-shell specimens were used in the experimental program. The first specimen, referred to as the preliminary shell, was used for instrumentation and test procedure checkout only. This specimen was made by cutting off the locally damaged region from the last shell tested in the sandwich shell buckling program (Reference 9), thus making it approximately 6.5 inches shorter than the final shell.

The second specimen, referred to hereafter as simply the shell, was made on the same mold as the other conical shells (Reference 9). However, in order to be able to handle the prepreg more easily during fabrication, the B-stage time was extended to 14 hours for the final specimen. This was double the B-stage time required to obtain optimal strength properties, but high-strength properties were not required in the present program. The resulting shell had considerably fewer wrinkles than those used in the buckling program.

Table I gives data on the sandwich consitutents, and Table II lists the dimensional and weight characteristics of the shell. The material properties of the facings and core are listed in Table III. The facing tensile properties and shear modulus were determined from tests on small tensile and torsion-tube specimens, which were subject to the same curing cycle as the facing laminates.

TABLE I. SANDWICH CONSTITUENTS

Constituent	Description	Thickness	Orientation
Facing	828-Z Epoxy, 181-E Volan A Fiber glass	Two-ply (0.02 inch total)	Warp parallel to shell axis
Core	5052 Aluminum, 1-mil Nonperforated Foil, 1/4-inch Hexagonal Cell	0.3 inch	Ribbon direction parallel to shell axis
Adhesive	AF-110B Film-Supported Epoxy	0.015 inch	-

TABLE II. FINAL SANDWICH SHELL DIMENSIONS AND WEIGHT	
Radius to middle of cross section (small end), R_1	22.45 inches
Radius to middle of cross section (large end), R_2	28.86 inches
Axial length	72.2 inches
Conical half angle, α	5.06 degrees
Nominal surface area of shell	11,680 square inches
Total weight of shell	55 pounds

TABLE III. SHELL-CONSTITUENT MECHANICAL PROPERTIES	
A. Facings (Average Test Data)	
Initial tension modulus, parallel to warp	$E_s = 3.64 \times 10^6$ psi
Initial tension modulus, perpendicular to warp	$E_\theta = 3.64 \times 10^6$ psi
Initial Poisson's ratio, loading parallel to warp	$\nu_{s\theta} = 0.20$
Initial Poisson's ratio, loading perpendicular to warp	$\nu_{\theta s} = 0.20$
Initial shear modulus, shearing plane perpendicular to warp	$G_{s\theta} = 1.33 \times 10^6$ psi
B. Core (Reference 38)	
Shear modulus, parallel to ribbon direction	0.0320×10^6 psi
Shear modulus, perpendicular to ribbon direction	0.0183×10^6 psi

3. EXPERIMENTAL EQUIPMENT

The shells were suspended from a large steel frame by six soft springs (Figure 1) so that the suspension resonant frequency was below 1 cps. The frame was previously used for casting molds (Reference 9). An electrodynamic exciter (MB Model C11, 50 pounds maximum force capacity) was attached to the shell by a force link that was designed to reduce the coupling between the specimen and the steel frame. The exciter was attached at different locations around the shell and along the length to determine the exciter effects on the data.

The instrumentation transducers used to measure the strain distribution at the resonant frequencies were 600 metallic-foil strain gages (Budd Model C6-141B) located on the outer facing of the shell in an array as shown on Figure 2. Two gages were located at each grid point, one to measure circumferential strain and one to measure meridional strain.

The electronic circuitry that was used to monitor and measure the dynamic signals from the gages is shown schematically in Figure 3, and a photograph is shown in Figure 1. The strain gage was used as a variable resistor, whose resistance change was a function of the elongation applied to the gage. A voltage of 7 volts D.C. was applied to the gage from a 300-volt D.C. power supply through a variable 10k-ohm dropping resistor. Analysis of the circuitry shows that not only was this a voltage-dropping resistor, but it was also a high impedance source to the A.C. signal coming from the strain gage. The signal was applied to the input of two A.C. amplifiers (Eico, Model 250) in series which had a high impedance to the D.C. portion of the signal, allowing only the A.C. component to be amplified. From the output of the amplifier, the signal was supplied to various instruments. The voltmeter (Hewlett-Packard Model 400H) and oscilloscope (Tektronix Model 503) were used to monitor the strain-gage amplitude and wave shape.

Each strain-gage system was calibrated separately. A known sinusoidal signal was applied across each strain gage as indicated in Figure 3. With a known input amplitude, the signal was recorded on the oscillograph. The 300-volt D.C. power supply was removed when the systems were being calibrated. With the system calibrated in this manner, errors in the absolute strain values were minimized.

The frequency response of the amplifier recording system has a flat response from 10 to 25,000 cps; the range of this experiment was from 5 to 500 cps. The responses of the strain gages for dynamic measurements were not measured in this test, but according to Reference 19, the responses are from 0 to 20,000 cps.

The equation for the strain measurements taken from the strain-gage system is (Reference 20):

$$\text{strain} = \varepsilon = (\Delta R'/R')/(G.F.)$$

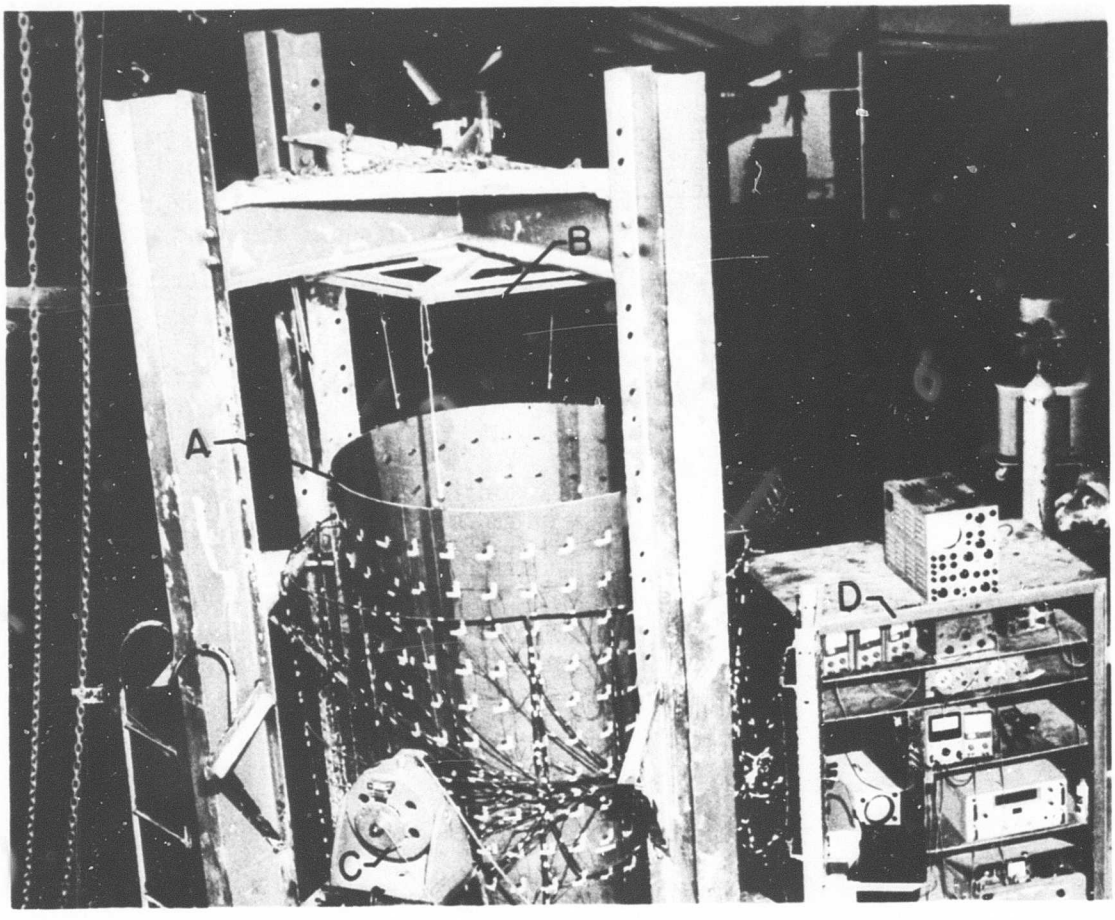
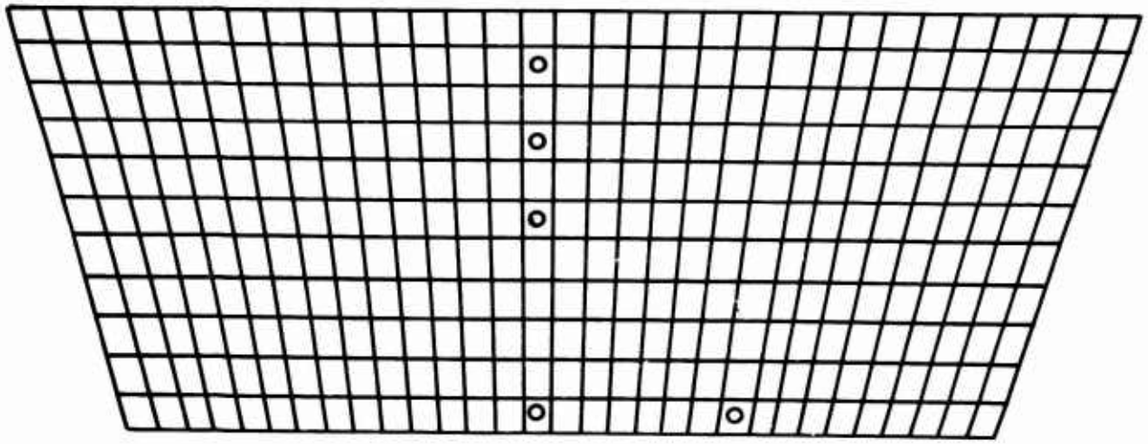


Figure 1. Experimental Vibration Test Setup (A, Conical Sandwich Shell; B, Soft Suspension Springs; C, Vibration Exciter; D. Instrumentation Cart).



O Exciter Locations

Figure 2. Layout of Shell Showing Strain-Gage Locations.

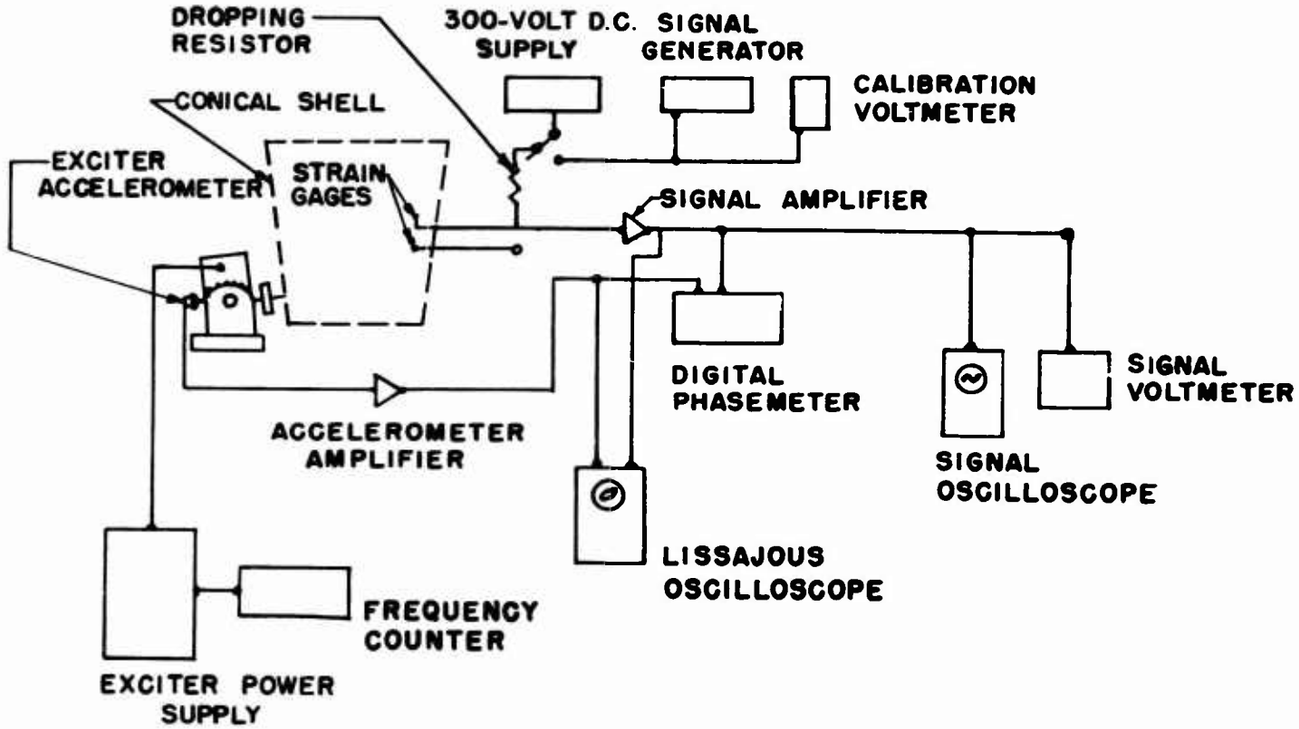


Figure 3. Electronic Instrumentation Schematic Wiring Diagram.

where G.F. = gage factor

$\Delta R'$ = change in gage resistance

R' = gage resistance.

Analysis of the electronic circuitry indicates that

$$(\Delta R'/R') = (\Delta V/V)$$

where ΔV = change in voltage across the gage

V = D.C. voltage across the gage.

Therefore,

$$\varepsilon_d = (\Delta V/V)(G.F.)$$

where ε_d = dynamic strain.

The voltage change was taken from the strain-gage output as noted on the voltmeter after applying the appropriate calibration factor. Substitution into the above equation gave the dynamic strain values.

An accelerometer (Endevco, Model FA-72) was attached to the armature of the vibration exciter to provide a measure of the input force into the shell. The output of the accelerometer (Endevco, Model 2016) was amplified and then was fed into a readout system, as indicated on Figure 3. The voltmeter was an indication of the output level, and the oscilloscope was used to monitor the signal to assure that there was no distortion.

The accelerometer system was calibrated by introducing a known signal in series with the accelerometer and by recording the output level. This was done in accordance with Reference 21.

The outputs of the strain-gage system and the accelerometer were applied to each side of the digital phase meter (AD-YU Digital Phase Meter, Model 524A and EAI Digital Voltmeter, Model 5002A), which indicated the relative phase between the input and strain distribution.

The system indicated on Figure 3 is for the output amplitude and phase of a single strain gage. Only one gage could be monitored at a time, since only one data acquisition system was available. A telephone switchboard was used to change gage locations. The switchboard was found to give the best signal-to-noise ratio at the lowest cost; other faster-switching arrangements were tried, but each increased the noise level. The strain-gage signals were on the order of 100 microvolts, which made the noise level of prime importance.

Other instruments of minor importance were included in the data acquisition

system. An oscillograph (Honeywell Visicorder, Model 906A) was used to record the log-decrement damping-factor signal. In this system, the signal was paralleled into the oscillograph from the voltmeter.

A stroboscope (General Radio Model 1531-AB) was used sometimes to help define the modal shape of the shell under excitation.

A second oscilloscope was used to display a Lissajous pattern as an indication of the phase between the input and strain. This was used as a check on the phase angle quadrant obtained from the phase meter.

A band-pass filter (SKL Model 302) was used sometimes to help eliminate the noise in the strain-gage signals. The filter was used sparingly, since it introduced a phase shift that was dependent on frequency. When the phase data were of prime importance, the filter was not used.

4. EXPERIMENTAL PROCEDURE

The procedure used to acquire the resonant-frequency and modal-strain-distribution data at each resonance was (1) to locate the resonant frequency using a modified Kennedy-Pancu technique (see the Appendix for development of this technique) and (2) to define the modal shape at this frequency.

The resonant frequency was located by monitoring a selected series of gages while the system was excited throughout the frequency range of interest. In this first survey, the peak amplitude method was used to locate the approximate range of the resonant frequency. The phase between the input and gage was not needed in this survey. The gages were selected by exciting the shell and by noting the antinodal points. Inherently, certain portions of the shell were excited at all the resonant frequencies. The survey was made by monitoring the input acceleration, strain outputs, and frequencies. Frequency intervals were taken at 5 cps across the entire band, with 1 cps near resonance. Closer intervals were taken to define the resonant point completely. Once the resonant frequencies were located, data were taken at these frequencies to make the modified Kennedy-Pancu plot.

The Kennedy-Pancu data were taken by varying the excitation frequency and by monitoring the phase angle between the input acceleration and the strain-gage signal. The gage selected for these data was the one that gave the largest output in the preliminary peak-amplitude survey. The excitation frequency was varied from a point below resonance that indicated near-zero phase to a point above resonance that indicated that the phase was returning to zero. Since the phase meter would not function below a threshold input voltage (0.35 volt), data were taken only in the general vicinity of resonances. These data were sufficient to define the uncoupled resonances by means of the characteristic circular arcs on the modified Kennedy-Pancu plots. To minimize the nonlinear effects of the shell, the strain level exhibited on the strain gage was held constant, and the input level into the shell was varied. The excitation frequency was changed until approximately 10-degree variation in phase was noted. The input force was varied until the gages exhibited a predetermined output; then, the input acceleration, output-gage signal, phase relation, and frequency were recorded.* This same procedure was repeated until the output signal became too small. The modified Kennedy-Pancu plot was drawn from these data, and the uncoupled resonant frequencies were determined. The excitation frequency of the shell was set at the uncoupled resonant frequency, and the excitation level was not changed from the previous data. Each strain-gage output and the corresponding phase relation were read. From these data, the modal strain distribution was determined. Once the data were acquired, the shell nodal points were investigated and checked by locating the points by feel and also by

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*At times, the phase meter would not give an accurate reading in relation to the angle's quadrant. The Lissajous pattern was used as a check to make sure that the quadrant was correct.

sight using the stroboscope.

The same procedure was repeated for each resonant frequency throughout the frequency range. Resonant frequencies were investigated from 5 to 400 cps. The lower limit was below the first resonant frequency and was the lowest frequency of the exciter. The upper limit was reached when the strain signal-to-noise ratio was too low to obtain reliable data.

5. ANALYSIS

a. Review of Previous Analyses

Although sandwich-type construction is coming into extensive use in aircraft structures in shell configurations, only a very limited range of vibrational problems involving such structures have been solved. Apparently, the first such analysis was performed by Yu (Reference 22) for the free vibrations of a simply supported complete cylinder of sandwich construction. This was followed by work on the following aspects for the same configuration: wave propagation by Chu (Reference 23), large-amplitude vibration by Chu (Reference 24), harmonic forced vibration by Bieniek and Freudenthal (Reference 25), structural damping by Yu (Reference 26) and Jones and Salerno (Reference 27), and effect of initial stress by Greenspon (Reference 28).

The cylindrically curved panel configuration, rather than a complete cylinder, was analyzed by Mead and Pretlove (Reference 29) and by Jacobson and Wenner (Reference 30). Shallow spherical-shell caps were analyzed by Tasi (Reference 31) and by Koplik and Yu (References 32, 33, and 34). Suvernev (Reference 35, as reported in Reference 36) treated conical-frustum shells.

None of the analyses mentioned above are applicable to sandwich shells with orthotropic facings, encountered in practice in the form of fiber-reinforced composites. However, some of them did take into consideration the much more simple effect of an orthotropic core, as characterized by the commonly used hexagonal-cell honeycomb core. The first sandwich-shell vibrational analyses to consider orthotropic facings were made independently by Azar (Reference 37), Vasitsyna (Reference 38, as reported in Reference 36), and Baker and Herrmann (Reference 39). Azar treated axisymmetric free vibrations of arbitrary open-ended shells of revolution, as exemplified by conical and paraboloidal shell frusta, with simply supported edges. Vasitsyna analyzed free vibrations of simply supported circular cylinders, while Baker and Herrmann considered the same problem with the addition of a general state of initial stress. Later Bacon and Bert (Reference 40) extended Azar's work to include unsymmetric modes.

It is important to mention that none of the analyses discussed above considered free-edge boundary conditions. Most of them used simply supported edges, and a few used clamped edges. As stated in the introduction, this is probably due to the increased mathematical complexity of the free-edge condition. In fact, only a very limited number of analyses of homogeneous, isotropic, conical shells with free edges have been carried out, and these only quite recently: Hu (Reference 41); Hu et al (Reference 42); Sewall, mentioned in a report by Mixson (Reference 17); and Krause (Reference 43).

Hu's analysis was formulated to include both membrane and bending

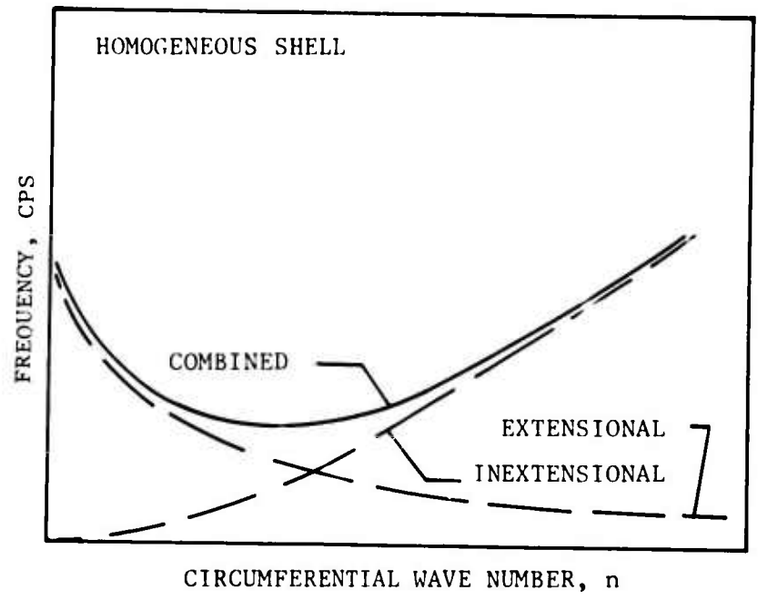
effects, with solution by the Galerkin method. However, the only numerical results which he published for free-free boundary conditions include only membrane effects, and they were not compared with any experimental results. The analysis by Hu et al was an inextensional analysis; i.e., one in which the membrane strains are identically zero, with solution by the Rayleigh-Ritz technique. Their results appeared to agree with their previously reported experimental results (Reference 18) for circumferential wave numbers, n, ranging from 2 to at least 9. Sewall's analysis was said to have been solved by a Rayleigh-Ritz technique, and results were presented for the lowest unsymmetric mode only. These results agreed quite well with experimental results reported by Mixson (Reference 17). Krause's analysis included both membrane and bending effects, and solution was accomplished by a modified Galerkin method in which it was not necessary that the assumed modal functions satisfy the force and moment boundary conditions. However, the results obtained by Krause did not appear to give better agreement with the results of Reference 18 than did the much more simplified analysis of Reference 42.

After a careful review of the analyses mentioned above, it was decided to use the Rayleigh-Ritz method of analysis, but to include the inextensional modes only. This choice was motivated by these practical considerations: (1) the best agreement reported to date for the unsymmetric vibrations of free-free, homogeneous, isotropic shells was obtained by the inextensional analysis of Hu et al (Reference 42); (2) in a sandwich shell, the magnitude of the membrane strain energy relative to the bending strain energy is considerably smaller than that for a homogeneous shell. (This is due to the difference in the ratio of membrane stiffness to flexural stiffness.) Thus, in a sandwich shell, the error introduced by neglecting membrane effects is small except for the axisymmetric case ($n = 0$). This is clearly shown by an approximate relation presented by Platus (Reference 44) for the natural frequencies, ω, of a shell:

$$\omega = (\omega_e^2 + \omega_i^2)^{1/2} \tag{1}$$

where ω_e is the frequency calculated by considering only extensional effects, and ω_i is the frequency calculated for the same n value but considering inextensional effects only. In deriving Equation (1), it is assumed that the modal functions associated with the two kinds of modes are geometrically similar, so that they affect the frequency only through their effect on the stiffness. The hypothesis that the extensional effects are relatively small for a sandwich conical shell were verified, for the case of freely supported edges, by the analysis of Bacon and Bert (Reference 40)(see Figure 4).

The analysis presented in the next section considers sandwich shells in the form of truncated cones with orthotropic facings and a perfectly rigid core. All components of translational inertia are included, but rotatory inertia is neglected.



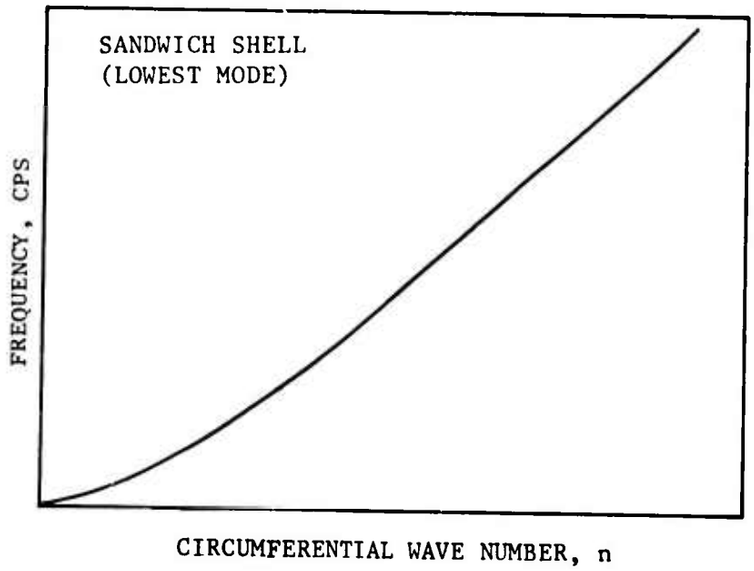


Figure 4. Effects of Inextensional and Extensional Deformations on Combined Frequencies for Free-Free Conical Shells of Homogeneous and Sandwich Types.

More refined analyses are currently in progress at the University of Oklahoma. These include membrane and bending effects in the orthotropic facings, transverse shear effects in the core, and all components of translational and rotatory inertia. Siu (Reference 45) is considering free-free boundary conditions and is using the energy method of solution. Wilkins (Reference 46) is using the Galerkin method and is including the following boundary conditions at both ends: freely supported, clamped, and free.

b. Analysis of Orthotropic Sandwich Conical Shell With Free Edges

Hu, Gormley, and Lindholm (Reference 42) presented an inextensional analysis of a homogeneous, isotropic, conical shell with free edges. Following the inextensional concept pioneered by Lord Rayleigh (Reference 42), they set the middle-surface strain components equal to zero and solved the resulting set of first-order differential equations to obtain the following expressions for the inextensional displacements:

$$
\begin{aligned}
u &= A \sin\alpha \cos\alpha \sin n\theta \cos\omega_i t \\
v &= (A + Bs/s_2)\, n \cos\alpha \cos n\theta \cos\omega_i t \\
w &= [A(n^2 - \sin^2\alpha) + Bn^2\, s/s_2] \sin n\theta \cos\omega_i t.
\end{aligned}
\tag{2}
$$

Here the same general approach as used by Hu et al is employed, except that an orthotropic sandwich conical shell is considered (see Figure 5). Since the extensional contribution to the strain energy is much smaller relative to the inextensional controbution in a sandwich shell than in a homogeneous one, an inextensional analysis is even more appropriate here than for a homogeneous shell (see Figure 4).

The homogeneous, linear algebraic equations for the constants A and B appearing in Equation (2) obtained from application of the Rayleigh-Ritz technique are as follows:

$$
\begin{aligned}
(\lambda^2 c_{11} - d_{11})\, A + (\lambda^2 c_{12} - d_{12})\, B &= 0 \\
(\lambda^2 c_{12} - d_{12})\, A + (\lambda^2 c_{22} - d_{22})\, B &= 0
\end{aligned}
\tag{3}
$$

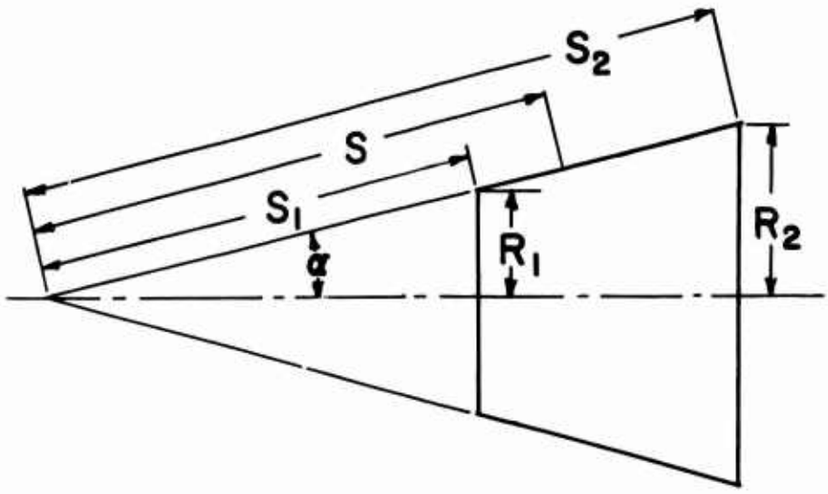
The eigenvalues, λ, for these equations are defined as follows:

$$\lambda = \omega_i R_2^2 (\rho h)^{1/2} [(1 + n^{-2} \cos^2\alpha)/(n^2 - 1) D_\theta]^{1/2} \tag{4}$$

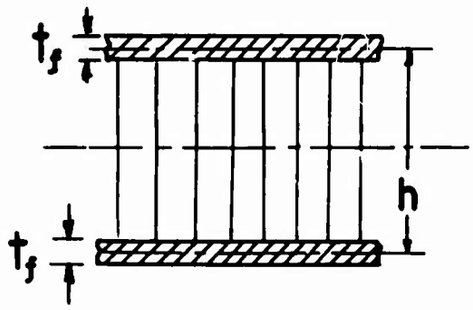
The solutions of Equations (3) are

$$\lambda = [Q \pm (Q^2 - 4PR)^{1/2}]/2P$$

and the corresponding ratios of B to A are



SHELL NOTATION



CROSS-SECTION NOTATION

Figure 5. Notation Used in Inextensional Analysis.

$$B/A = (\lambda^2 c_{11} - d_{11})/(\lambda^2 c_{12} - d_{12}) \tag{5}$$

where

$$P = c_{11} c_{22} - c_{12}^2$$

$$Q = c_{11} d_{22} + c_{22} d_{11} - 2c_{12} d_{12} \tag{6}$$

$$R = d_{11} d_{22} - d_{12}^2$$

The values for the factors in Equations (6) are

$$c_{11} = [(1 - \gamma^2)/2]\{1 - [(2n^2 - 1) \sin^2 \alpha]/[n^2(n^2 + \cos^2 \alpha)]\}$$

$$c_{12} = [(1 - \gamma^3)/3][1 - (\sin^2 \alpha)/(n^2 + \cos^2 \alpha)]$$

$$c_{22} = (1 - \gamma^4)/4 \tag{7}$$

$$d_{11} = \{[(1/\gamma^2) - 1]/2\}\{1 + (D_{s\theta}/D_\theta)[(\sin \alpha)/n^2]$$

$$d_{12} = (\gamma^{-1} - 1)$$

and

$$d_{22} = \log_e (1/\gamma)$$

where γ is the completeness parameter (R_1/R_2), and for a sandwich with thin facings,

$$D_\theta = E_\theta t_f h^2/[2(1 - \nu_{s\theta} \nu_{\theta s})] \tag{8}$$

and

$$D_{s\theta} = 2G_{s\theta} t_f h^2 \tag{9}$$

Equations (7) are identical to those given by Hu et al, except the expression for d_{11}. However, for the isotropic case considered by Hu et al, $D_{s\theta}/D_\theta$ becomes $2(1-\nu)$, and the expression for d_{11} coincides with that given by Hu et al.

When the expression for the B/A ratio, Equation (5), is substituted into the expression for the circumferential displacement, the second of Equations (2), the resulting expression for the normalized modal shape is

$$V = \{1 + [(\lambda^2 c_{11} - d_{11})/(\lambda^2 c_{12} - d_{12})](s/s_2)\} \tag{10}$$

Similarly, when Equation (5) is substituted into the expression for normal displacement, the third of Equations (2), the resulting expression for the normalized modal shape is

$$W = n^2 \{1 - (\sin^2 \alpha)/n^2 + [(\lambda^2 c_{11} - d_{11})/(\lambda^2 c_{12} - d_{12})]s/s_2\} \tag{11}$$

Since the middle surface extensional strains and the meridional bending strain are zero, the only strain existing is the circumferential bending strain, $\varepsilon_{b\theta}$. The circumferential surface bending strain is

$$\varepsilon_{b\theta} = [(h + t_f)/2]\ \kappa_\theta \tag{12}$$

where κ_θ is the change in circumferential curvature.

Thus, the meridional distribution of the dimensionless circumferential surface bending strain is

$$\bar{\varepsilon}_{b\theta} = [(s_1/s)^2 + (B/A)(s_1^2/s_2 s)] \tag{13}$$

Although the results of this simple analysis are in algebraic form, for the values of geometric and material parameters involved, subtraction of two large and nearly equal numbers is involved. Therefore, to obtain reasonable numerical results, the above equations for the eigenvalues, eigenvectors, modal shape, and strain distribution were programmed in FORTRAN IV, double precision language and were computed on the University of Oklahoma's IBM 360/40 digital computer. The program was written to obtain the various quantities as a function of the circumferential wave number, n, with the semi-vertex cone angle, α, as a variable.

6. EXPERIMENTAL AND ANALYTICAL RESULTS

The experimentally measured values of resonant frequencies and associated strain distributions and damping are presented in this section. The resonant frequencies and associated strain distributions for the two lowest unsymmetric modes are compared with the analyses presented in Section 5.

Resonant frequencies for various meridional modes are shown in Figure 6 as a function of circumferential mode number. On this figure, the two curves that were derived from the inextensional analysis presented in Section 5 are shown along with the experimental resonant frequency points. Each resonant point was obtained from a modified Kennedy-Pancu plot. Some typical plots that were used to separate the resonant frequencies are shown in Figures 7 through 14. As can be seen from the figures, the separation of the resonant frequencies that were very close together was accomplished quite adequately by means of the Kennedy-Pancu technique. Damping factors were determined from these figures using the technique outlined in the appendix.

It can be seen in Figure 6 that the experimentally measured resonant-frequency points associated with the two lowest unsymmetric modes agree quite closely with the frequencies calculated by the inextensional analysis over the entire range of circumferential wave numbers covered.

The resonant-frequency curves for the higher modes indicate the same trend as those of the two lowest modes. In the region of low circumferential wave numbers, n, the higher modes were difficult to detect because the resonant frequency coincided with the resonant frequency of another mode. At these lower resonant frequencies, the response of the mode having a higher n would dominate that of the lower-n mode. This problem was one of the major difficulties encountered in applying the Kennedy-Pancu technique in this investigation. It was beyond the scope of this investigation because of the limitation of equipment, but a solution to this problem would be to introduce additional excitation systems. By placing the exciters near a point that is a node of the strong mode, the system would exhibit a larger response in the weak mode.

Also, the higher modes displayed the same trend as the lower modes at the higher circumferential wave numbers and indicated a possible convergence of all the modes, if higher wave numbers and frequencies could have been obtained.

At the lower wave numbers, it is believed that the higher modes will not indicate as severe of a hook, similar to the curves presented by Watkins and Clary (Reference 47), Hu et al (Reference 18), and others in reporting the unsymmetrical resonant frequencies of homogeneous-material conical shells. The reason for this was discussed in Section 5 in connection with Equation (1).

Shown as Figures 15 and 16 are the plots of circumferential strain associated

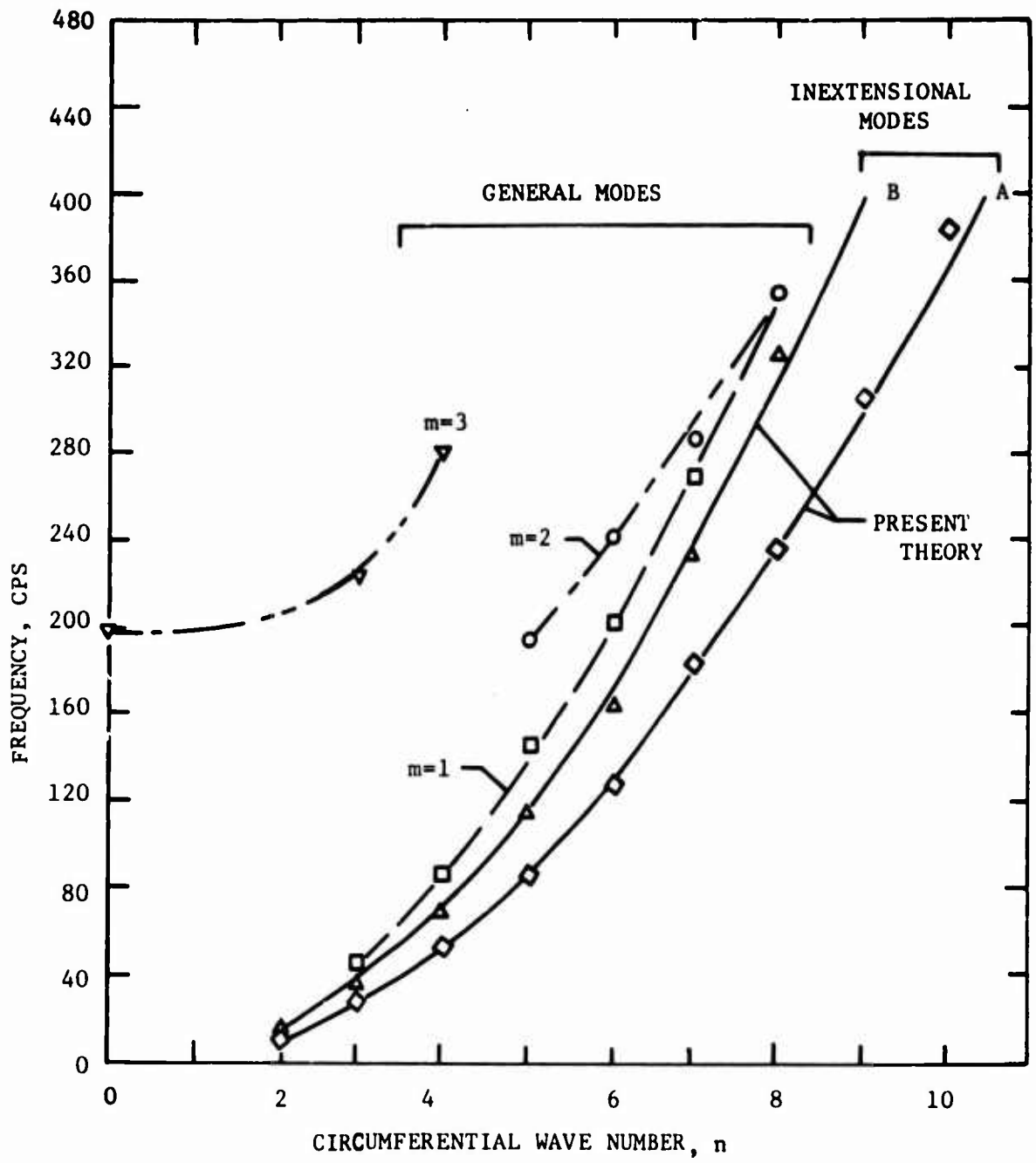


Figure 6. Experimental and Analytical Resonant Frequency Variations.

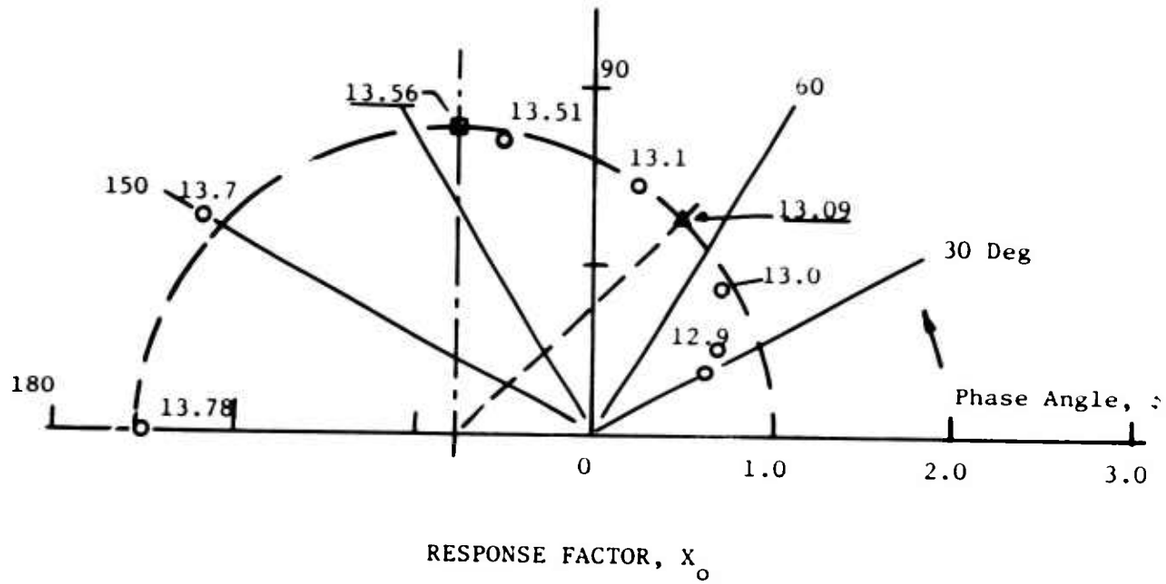


Figure 7. Kennedy-Pancu Plot for 13.56 cps.

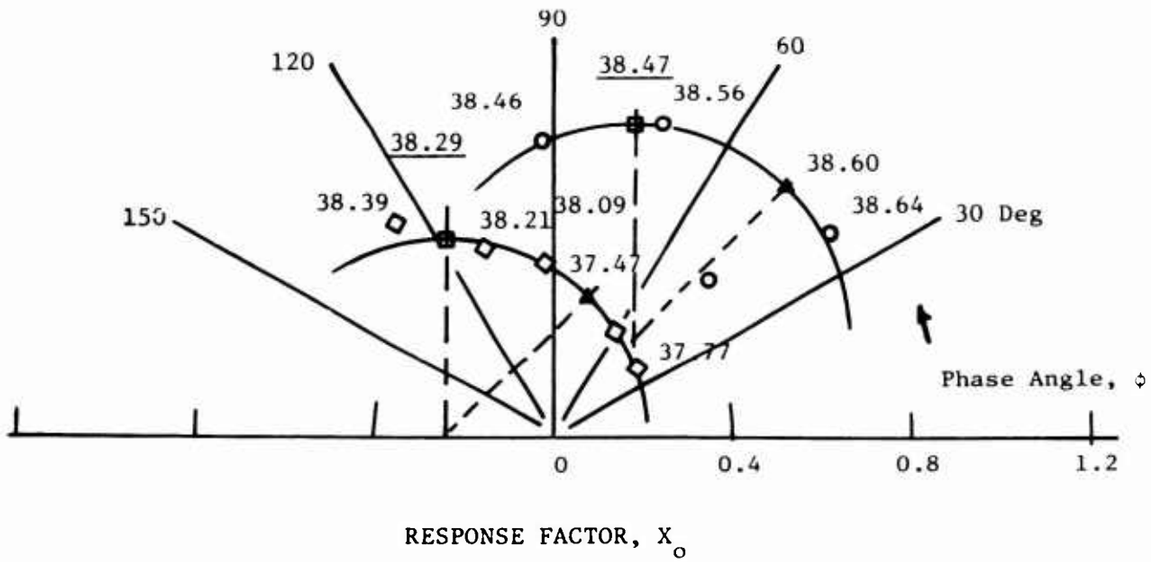


Figure 8. Kennedy-Pancu Plot for 38.47 and 38.29 cps.

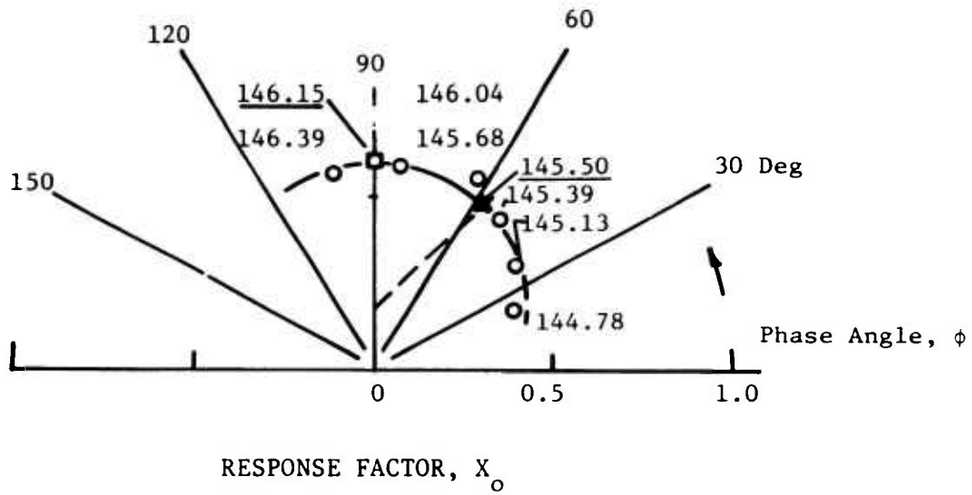


Figure 9. Kennedy-Pancu Plot for 146.15 cps.

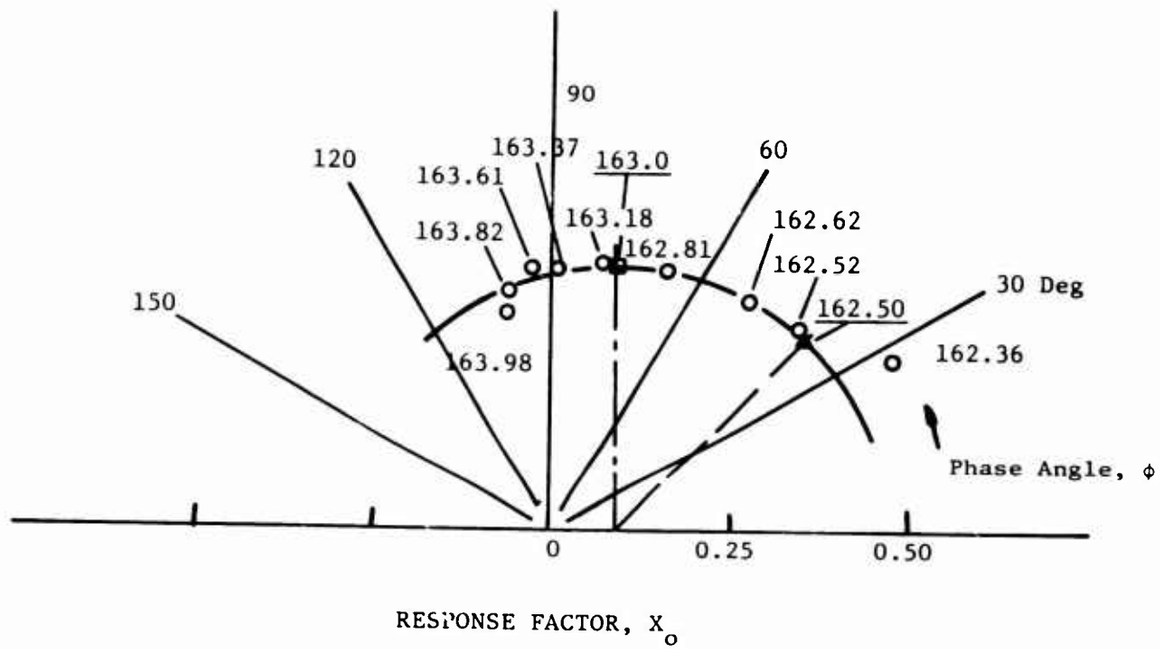


Figure 10. Kennedy-Pancu Plot for 163.0 cps.

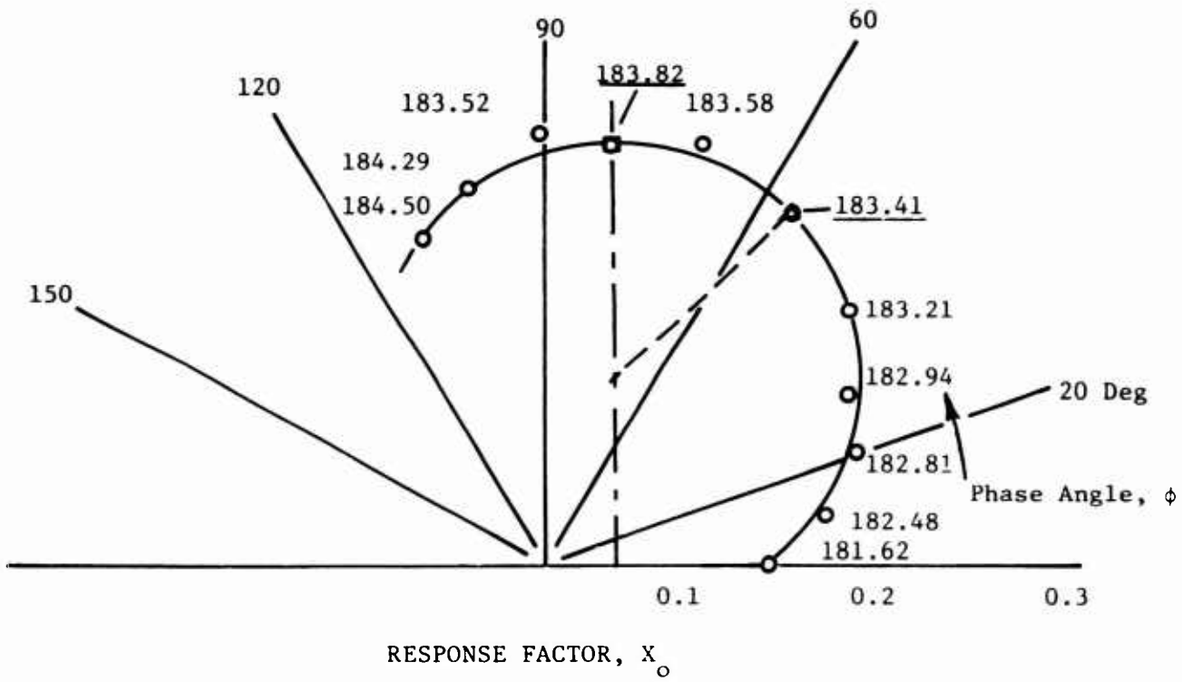


Figure 11. Kennedy-Pancu Plot for 183.8 cps.

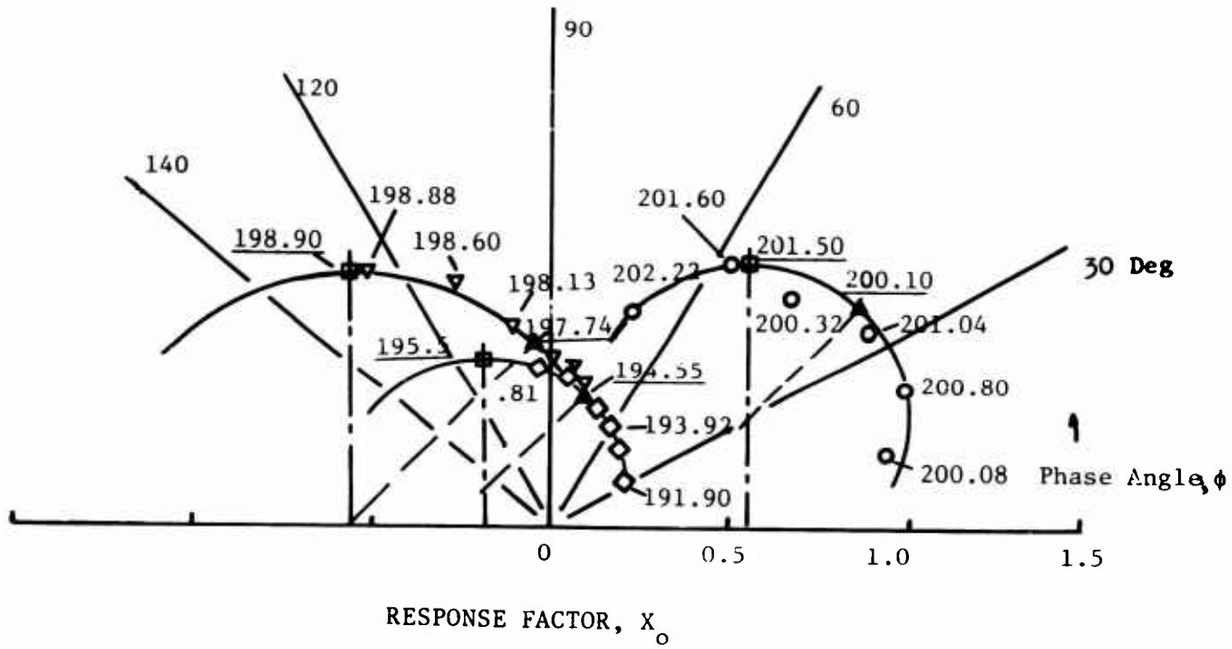


Figure 12. Kennedy-Pancu Plot for 195.5, 198.9, and 201.5 cps.

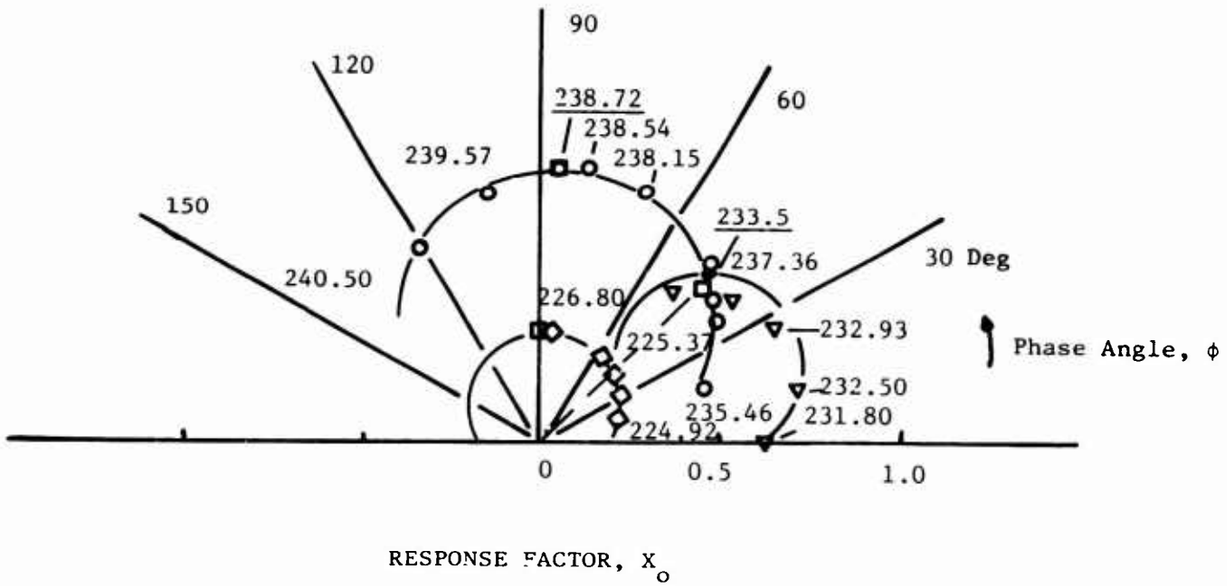


Figure 13. Kennedy-Pancu Plot for 226.8, 233.5, and 238.7 cps.

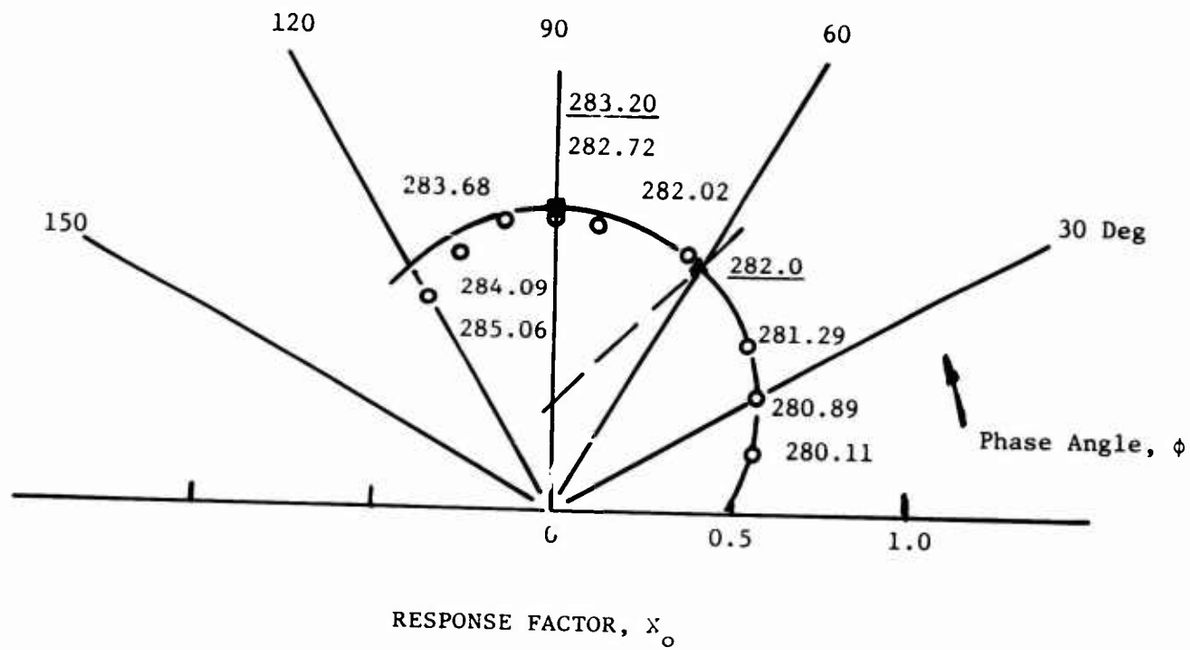


Figure 14. Kennedy-Pancu Plot for 283.2 cps.

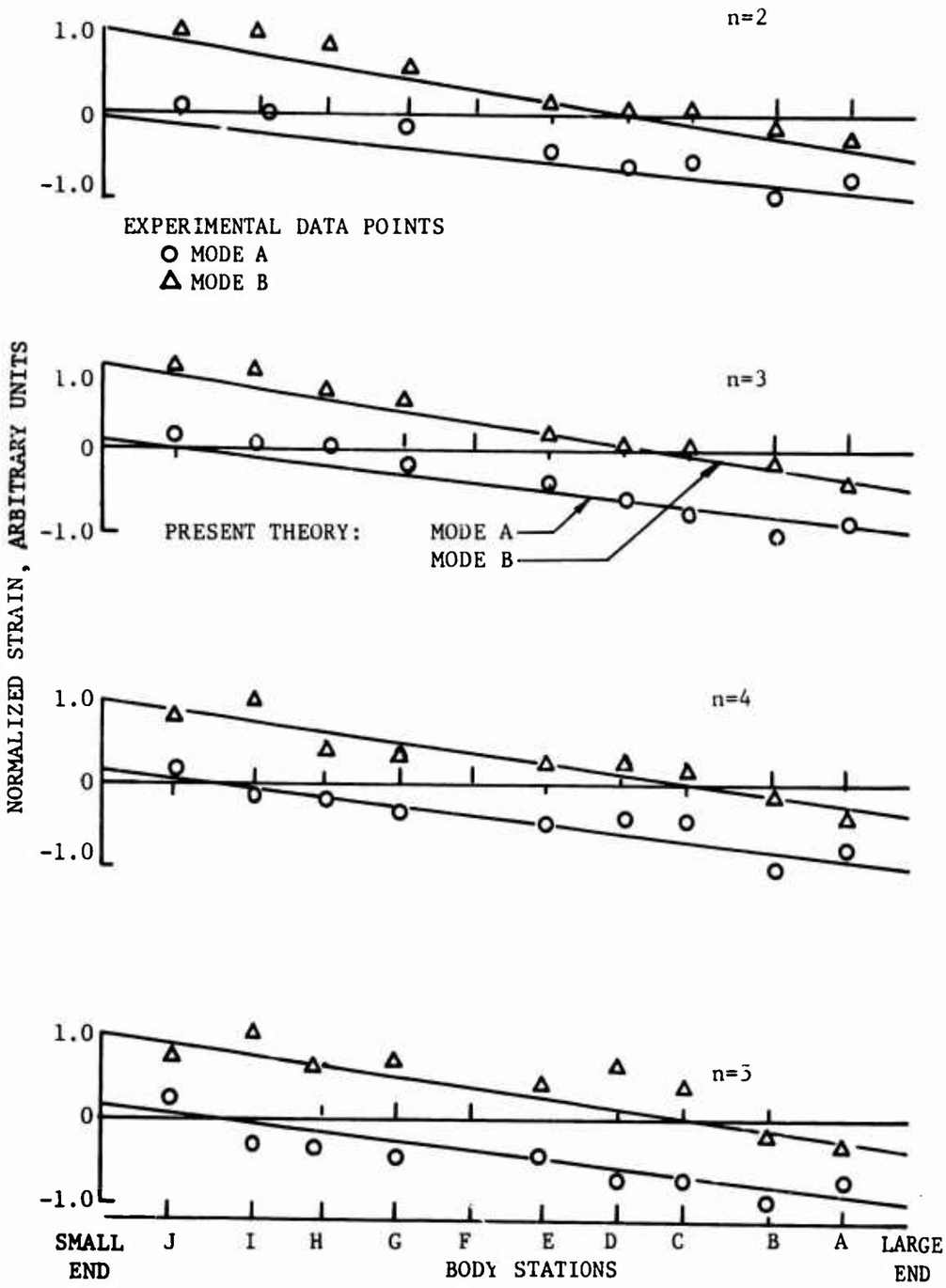


Figure 15. Normalized Circumferential Strain Distribution Along Meridian, Inextensional Modes, n = 2 through 5.

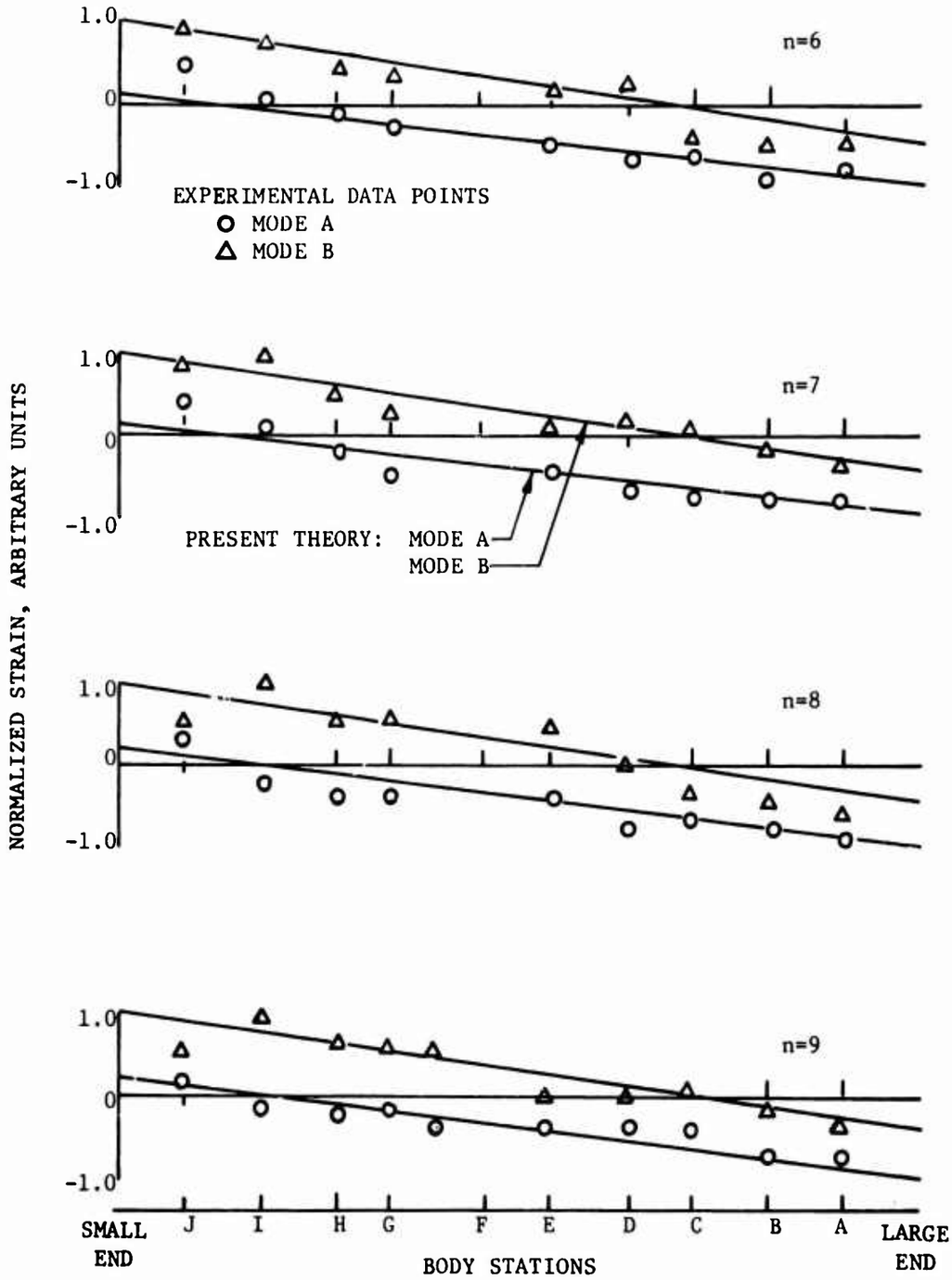


Figure 16. Normalized Circumferential Strain Distribution Along Meridian, Inextensional Modes, n = 6 through 9.

with the two lowest meridional modes. Also included as Figures 15 and 16 are the analytical, circumferential strain distributions derived from the inextensional analysis presented in Section 5. In general, there is very good agreement between the analytical and experimental strain distributions for these modes. The effect of exciter mass and location is noticeable in the meridional modal shapes. However, this is to be expected since it was observed by Mixson (Reference 17). The circumferential modal shapes are symmetrical about the exciter location. Moving the exciter to a new circumferential location on the shell resulted in the same modal shape but with the nodal points shifted circumferentially.

Figure 17 shows a plot of the meridional strain distribution associated with the two lowest meridional modes. The meridional strain distribution is identically zero in the inextensional analysis presented in Section 5. Even though the curves indicate a meridional strain, the values are small and can be attributed to the shell's being restrained from completely free movement at the shaker-attachment location and at the suspension points.

Since the experimental data and the values derived by the inextensional analysis are in close agreement, it is assumed that the lowest experimental unsymmetrical modes are the inextensional modes. The agreement is close for:

(a) the resonant frequencies as a function of circumferential wave number,

(b) the circumferential strain as a function of meridional position,

(c) the meridional strain as a function of meridional position.

Hereafter, these two lowest experimental modes will be referred to as the inextensional modes, since these agree closely with the analytical inextensional modes. The other modes which have higher resonant frequencies will be referred to as general modes, since they exhibit both extensional and inextensional deformation as discussed in Section 5.

The circumferential strain distribution patterns associated with the circumferential modes for the inextensional and general modes are shown in Figures 18 and 19. The circumferential strain distributions associated with the general modes are shown in Figure 20. The distributions of meridional strain along the meridian are shown in Figure 21. It can be noted from this figure that the meridional strain is not uniform along the meridian, in contrast to the cases of the two inextensional modes. It can be seen in Figure 6 that the resonant frequency for the m = 1 general mode is close to inextensional mode B at the low circumferential wave numbers, but as the wave number increases, the curves separate and the m = 1 mode converges to the m = 2 general mode. Also, it can be noted in this figure that all of the general modes tend to converge at the higher frequencies. This convergence is characteristic of the dynamics of both cylindrical and conical shells.

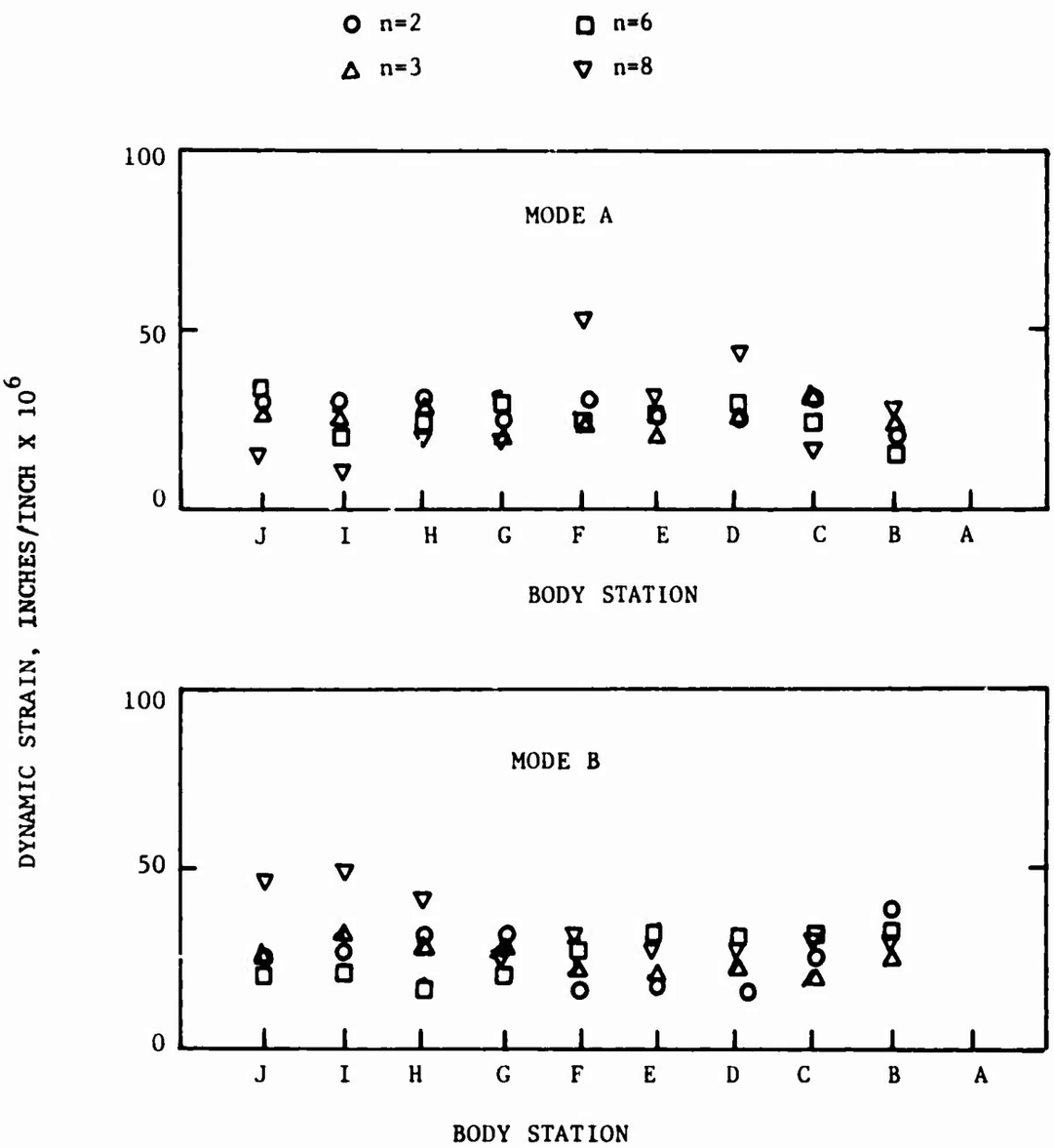


Figure 17. Meridional Strain Distribution Along Meridian, Inextensional Modes, n = 2, 3, 6, 8.

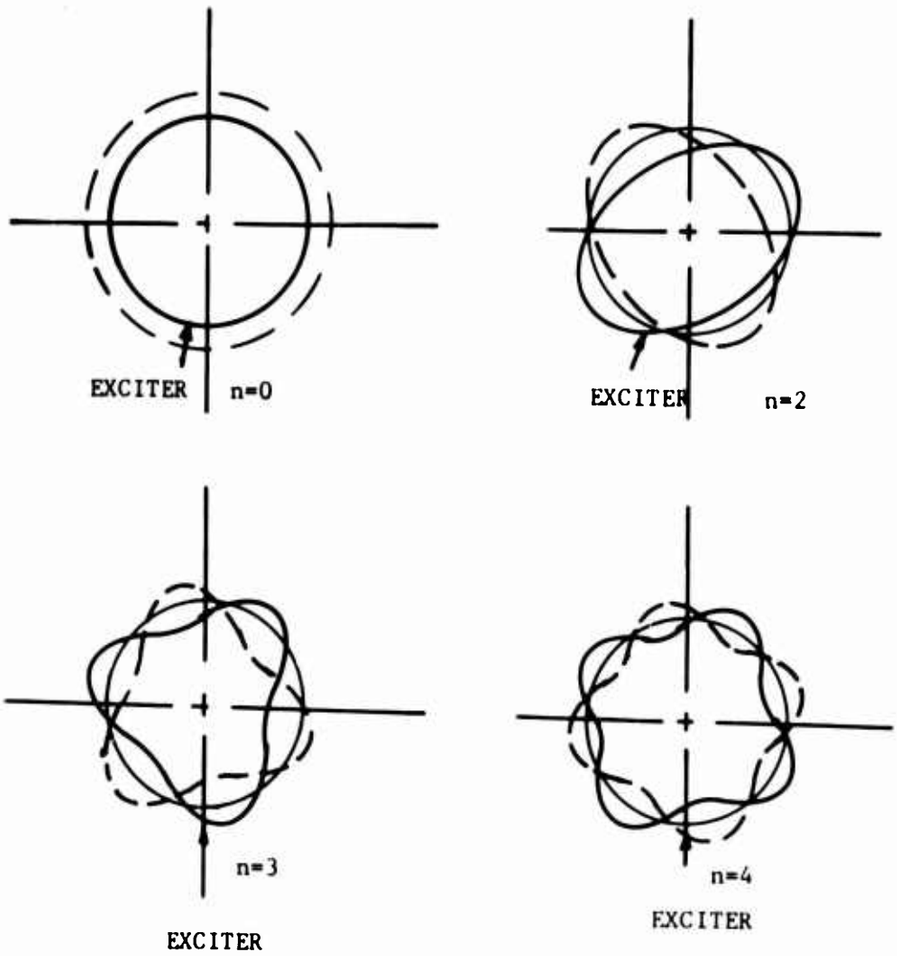


Figure 18. Cross Section of Circumferential Waveforms, n=0, 2, 3, 4.

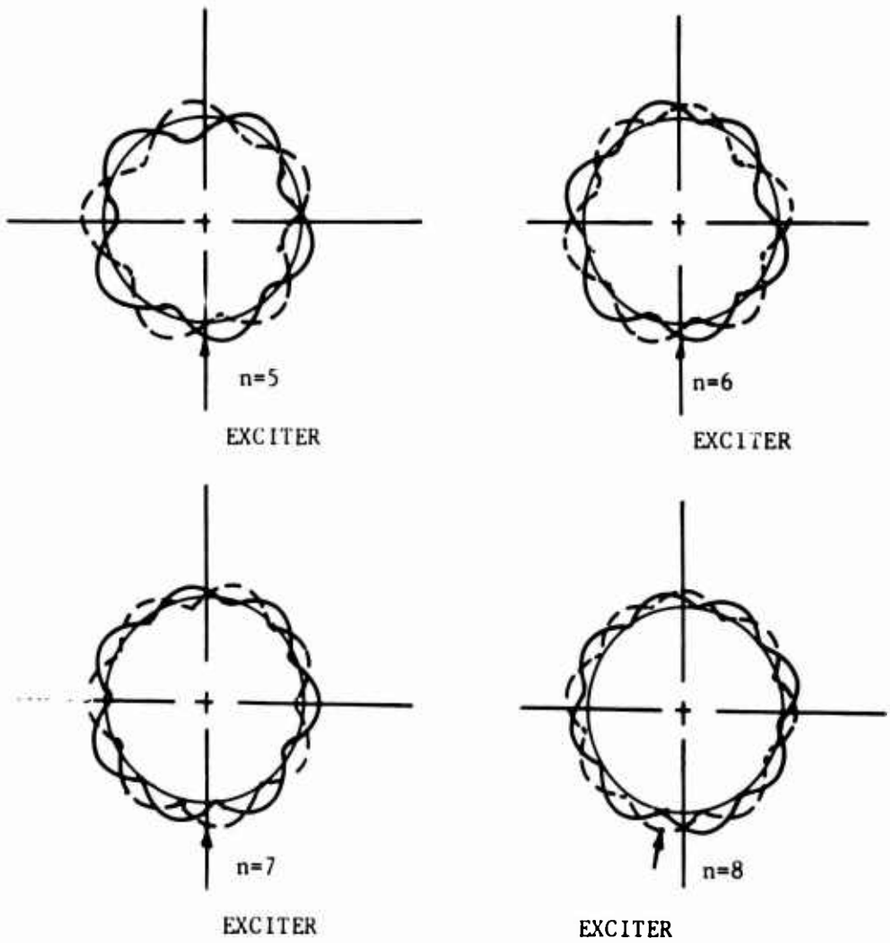


Figure 19. Cross Section of Circumferential Waveforms, n=5, 6, 7, 8.

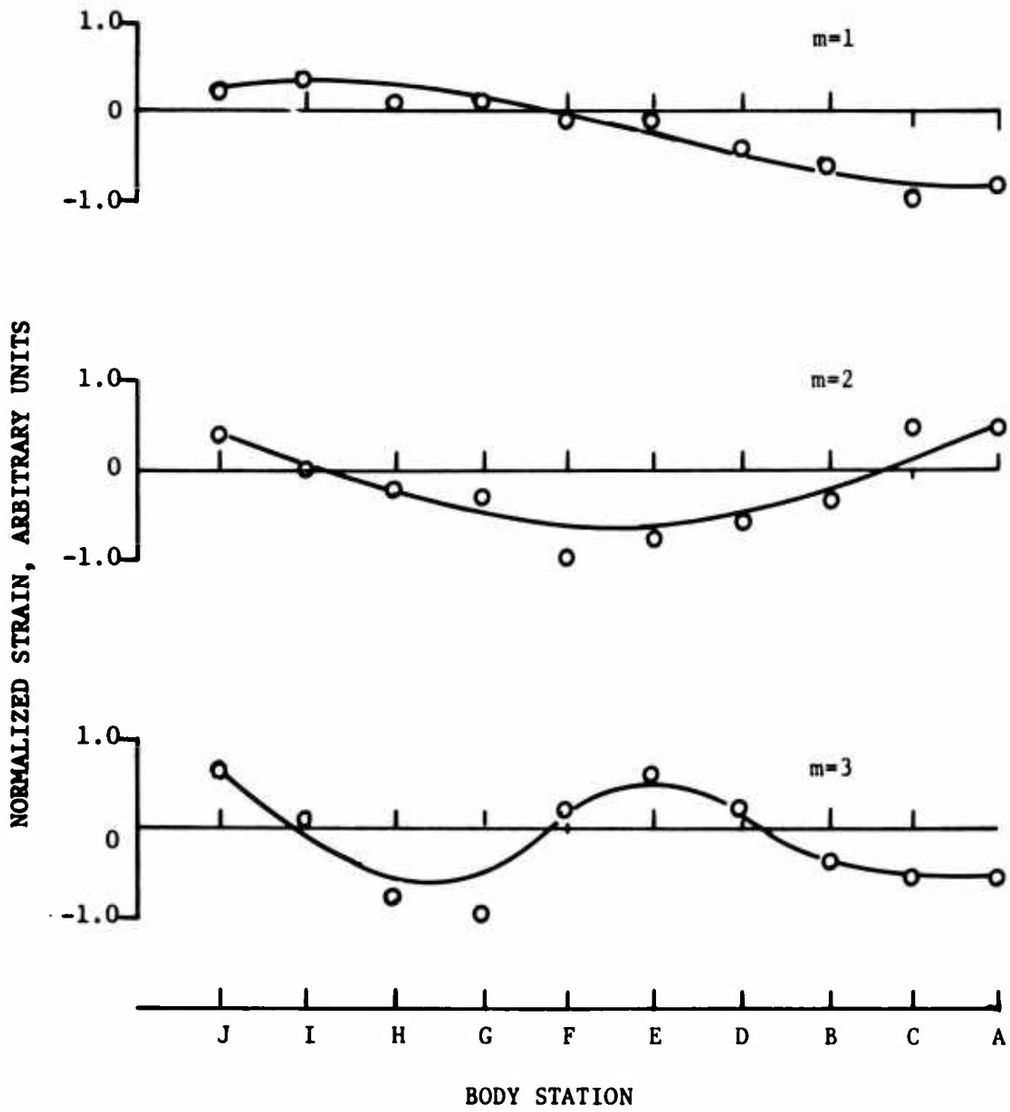


Figure 20. Normalized Circumferential Strain Distribution Along Meridian, General Modes.

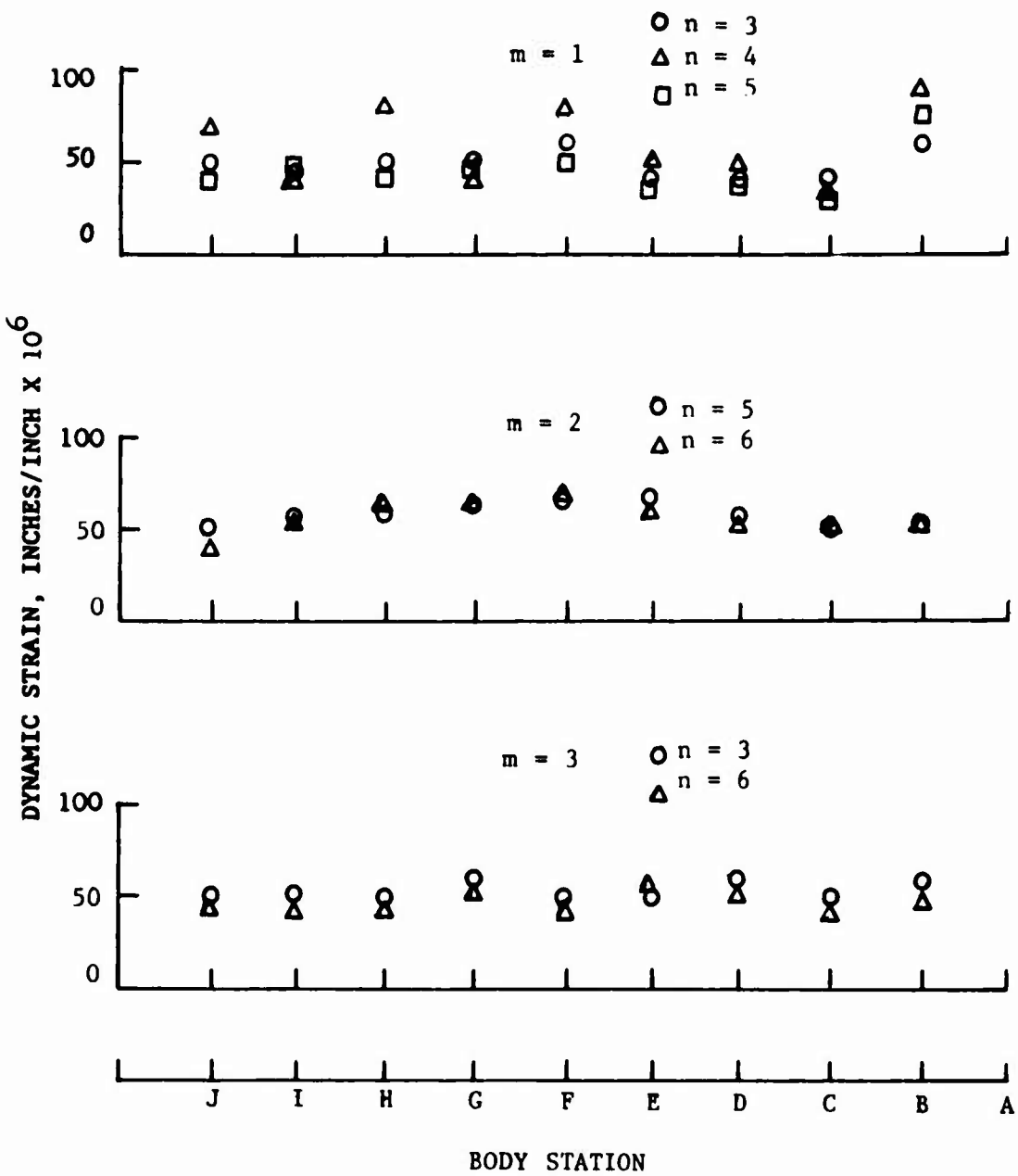


Figure 21. Meridional Strain Distribution Along Meridian, General Modes.

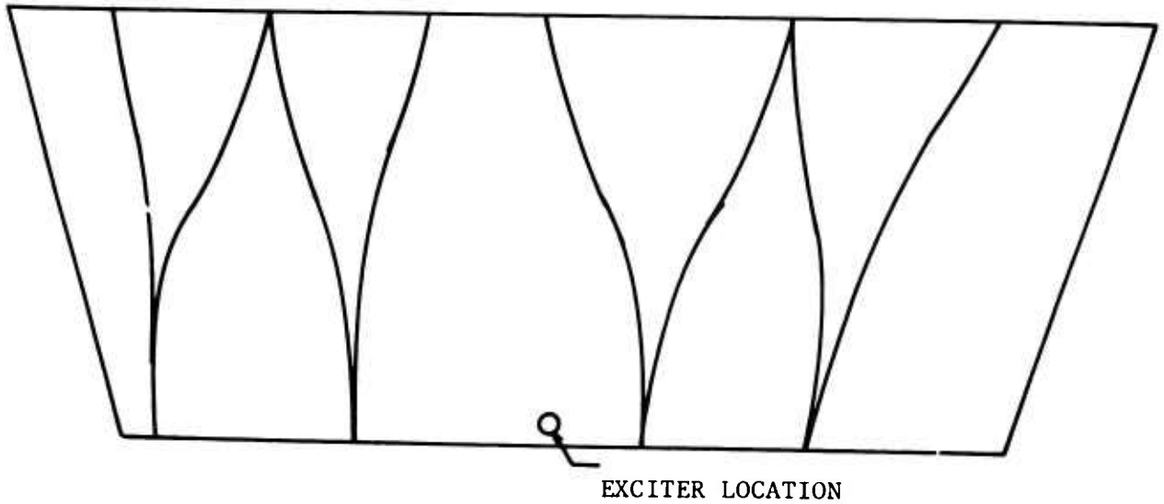
The resonant-frequency data and the associated strain distributions indicated that some coupling between modes occurred. These coupled modes were a combination of two circumferential modes having the same meridional mode when the resonant frequencies were close.

Pronounced coupling phenomena were observed at resonant frequencies of 33 cps, 64 cps, and 94 cps. In all of these cases, the strain distributions were typically the same except for the difference in circumferential wave numbers. Shown as Figures 22 through 24 are typical plots of the nodal patterns associated with the above frequencies. In each of these figures, the nodal patterns were symmetrical about the exciter but not evenly spaced.

The phenomenon of U-shaped nodal lines, i.e., the existence of more circumferential waves at the large end of a free-free conical-frustum shell than at the small end, was observed in experiments reported by Watkins and Clary (Reference 16). This has resulted in considerable controversy, since they were not observed at all in any of the similar experiments reported by Hu et al (Reference 18). Hu (Reference 48) suggested that the U-shaped modes were caused by shaker effects, since his experiments (Reference 18) were conducted by magnetic fields (no direct contact with shell). Additional comments were made by Koval (References 49 and 50), who showed photographs of similar phenomena in vibrational experiments on cylindrical shell. He attributed the U-shaped-mode phenomenon to the dynamic asymmetries caused by the spot-welded longitudinal seams present in Watkins and Clary's shells. According to the theory originated by Tobias (Reference 51) and applied by Arnold and Warburton (Reference 52), the presence of small imperfections causes each of the two close resonant frequencies to have a preferential nodal pattern. Then, under excitation at a frequency between the two component frequencies, the two different component modes combine to produce the complicated mixed-mode pattern.

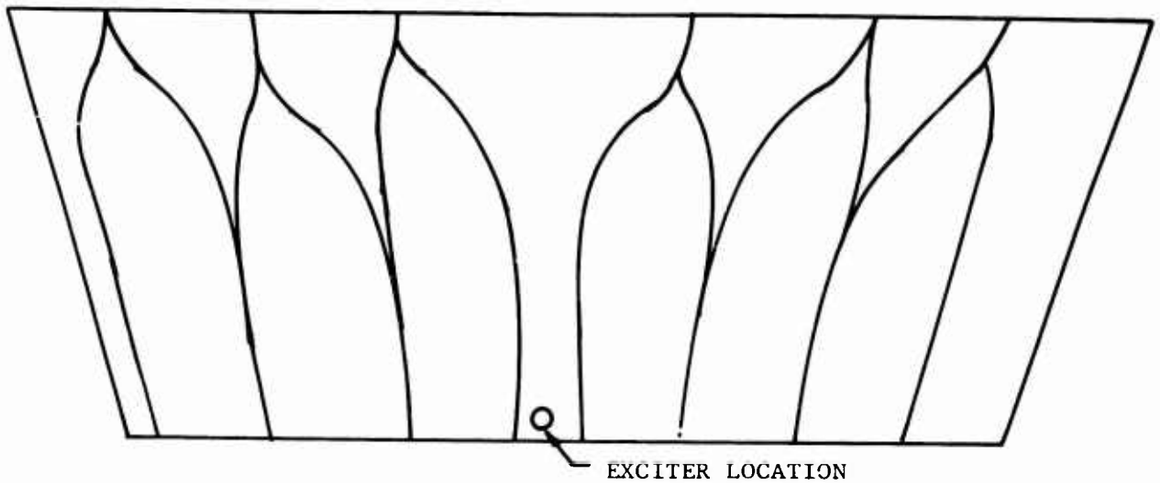
As a result of the controversy, following the suggestion of Watkins and Clary (Reference 47), Mixson (References 17 and 53) conducted an extensive series of additional experiments on shells with and without longitudinal seams. In these experiments, he used air shakers, as well as electrodynamic ones, and varied the ratio of the frequency of the axial rigid-body mode (controlled by the suspension stiffness) to that of the lowest elastic shell mode. He concluded that imperfections were probably the principal cause of the mixed modes, since they occurred most often in the cones with seams and smaller wall thickness (in which it is more difficult to control imperfections).

The coupling observed was probably caused by the presence of the four equally spaced longitudinal lap joints on the inside and outside facing (i.e., a total of eight lap joints; one every 45 degrees). However, it is interesting to note that the only modes which coupled were inextensional modes (A and B). There was no observed coupling of the general modes ($m = 1, 2, 3$) with each other or with the inextensional modes. This may have been because, for the particular shell construction and geometry, the inextensional modes were the only ones sufficiently close together (in



SHELL LAYOUT

Figure 22. Nodal Pattern for Coupled Mode at 33 cps.



SHELL LAYOUT

Figure 23. Nodal Pattern for Coupled Mode at 64 cps.

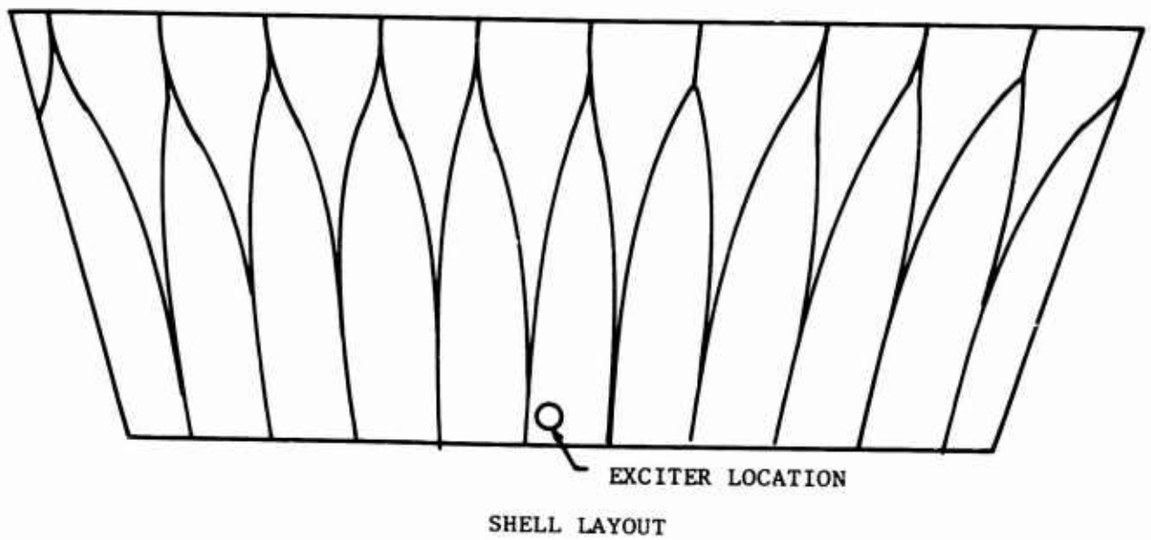


Figure 24. Nodal Pattern for Coupled Mode at 94 cps.

frequency as well as in circumferential wave number) to couple.

At present, the authors know of no analytically based criterion for determining a priori whether or not two adjacent modes will couple to form a combined mode. Since the available experimental data are quite limited and expensive to obtain, an analytically based criterion, rather than an empirically based one, would be quite useful.

Since the conical shell used here had a small cone angle, it could be approximated, with reasonable accuracy, by a cylinder for calculating the frequencies of certain higher modes peculiar to sandwich structures. For these calculations Yu's analysis (Reference 22) was used. It was found that the calculated frequencies for all three of these modes (shear modes in both the meridional and circumferential planes and the thickness-normal mode) were much higher than the highest frequencies attained in the experiments. Therefore, it is presumed that no modes of these types were excited in the experiments reported.

Figure 25 shows a plot of the damping logarithmic decrement as a function of resonant frequency. The values for this figure were obtained from the Kennedy-Pancu plots and converted to logarithmic decrements. As pointed out in the appendix, the damping coefficient of the system can be obtained from the Kennedy-Pancu plots. The damping coefficient is converted to logarithmic decrement by multiplying the coefficient by 2π. The tendency for the damping to decrease as the frequency increases agrees with data presented by Shoua (Reference 54). The damping logarithmic decrement at the higher frequencies agrees with values presented by Shoua and Bert et al Reference 8) for beams of the same type of construction and materials as used in the present shells (sandwich with fiber glass epoxy facings and aluminum honeycomb core).

The same damping logarithmic decrement is shown in Figure 26 as a function of circumferential wave number.

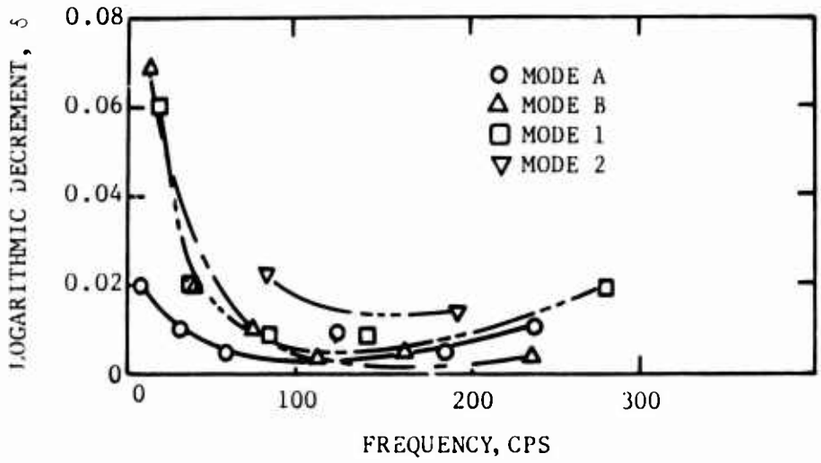


Figure 25. Damping Logarithmic Decrement as a Function of Resonant Frequency.

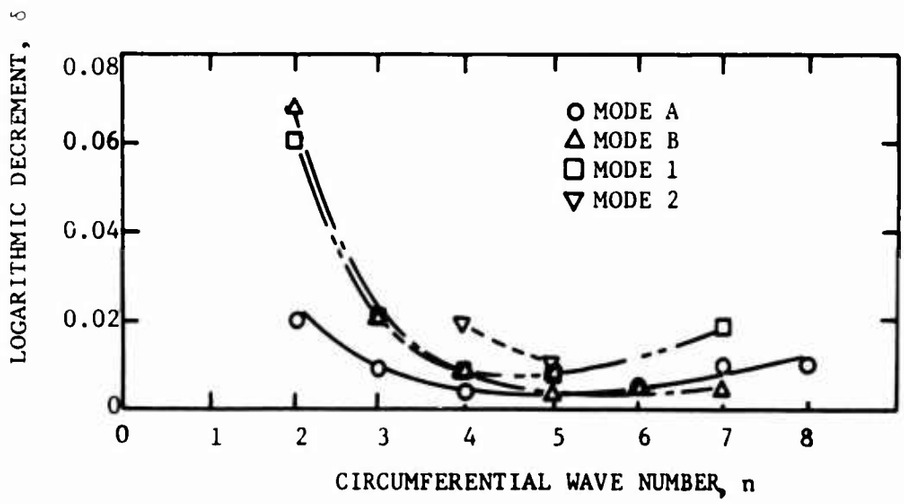


Figure 26. Damping Logarithmic Decrement as a Function of Circumferential Wave Number.

7. CONCLUSIONS AND RECOMMENDATIONS

a. Conclusions

(1) With sufficiently accurate electronic instrumentation now available commercially and with sufficient care in conducting the experiments, a modified Kennedy-Pancu technique has been applied successfully to shell-type structures to determine very accurately the resonant frequencies, modal strain distributions, and damping associated with pure modes.

(2) For free-free sandwich shells, an inextensional analysis predicts quite accurately the resonant frequencies associated with the first two unsymmetric modes. Under the conditions of the present experiments, extensional,core shear, and core normal effects can be neglected for these two modes.

(3) The combined modes with U-shaped nodal patterns, as reported in previous experiments on free-free cones, were observed. However, in all cases, they represented coupling between inextensional modes only.

b. Recommendations

(1) For applications involving very high frequency excitations, such as blast loading or high-frequency sonic excitation, the higher order effects of transverse shear deformation, thickness normal deformation, and rotatory inertia may be important. To investigate higher frequencies, additional research, using the same size of sandwich shells and larger excitation force, would be required.

(2) An analysis is needed to predict a priori whether or not two adjacent modes will or will not couple into combined modes.

LITERATURE CITED

1. Nordby, G.M., and W.C. Crisman, RESEARCH IN THE FIELD OF FIBERGLASS-REINFORCED SANDWICH STRUCTURE FOR AIRFRAME USE, University of Oklahoma Research Institute; TRECOM Technical Report 64-37, U.S. Army Transportation Research Command, Fort Eustis, Virginia, July 1964, AD 607339.

2. Nordby, G.M., and W.C. Crisman, FABRICATING LAMINATES FOR SPECIFIC PROPERTIES, Modern Plastics, Vol. 44, No. 2, October 1966, pp. 135-136.

3. Nordby, G.M., W.C. Crisman, and C.W. Bert, THE EFFECT OF RESIN CONTENT AND VOIDS ON THE STRENGTH OF FIBERGLASS-REINFORCED PLASTICS FOR AIRFRAME USE, University of Oklahoma Research Institute; USAAVLABS Technical Report 65-66, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, November 1965, AD 627362.

4. Harris, B.J., and W.C. Crisman, FACE WRINKLING MODE OF BUCKLING OF SANDWICH PANELS, Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 91, No. EM3, June 1965, pp. 93-111.

5. Nordby, G.M., and W.C. Crisman, STRENGTH PROPERTIES AND RELATIONSHIPS ASSOCIATED WITH VARIOUS TYPES OF FIBERGLASS-REINFORCED FACING SANDWICH STRUCTURE, University of Oklahoma Research Institute; USAAVLABS Technical Report 65-17, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, August 1965, AD 616617.

6. Nordby, G.M., W.C. Crisman, and C.W. Bert, DYNAMIC ELASTIC, DAMPING, AND FATIGUE CHARACTERISTICS OF FIBERGLASS-REINFORCED SANDWICH STRUCTURE, University of Oklahoma Research Institute; USAAVLABS Technical Report 65-60, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, October 1965, AD 623128.

7. Nordby, G.M., and W.C. Crisman, FATIGUE CHARACTERISTICS OF AN RP SANDWICH STRUCTURE, Modern Plastics, Vol. 43, No. 10, June 1966, pp. 120 ff.

8. Bert, C.W., D.J. Wilkins, Jr., and W.C. Crisman, DAMPING IN SANDWICH BEAMS WITH SHEAR FLEXIBLE CORES, Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers, Vol. 89B, No. 4, November 1967, pp. 662-670.

9. Nordby, G.M., W.C. Crisman, and C.W. Bert, FABRICATION AND FULL-SCALE STRUCTURAL EVALUATION OF SANDWICH SHELLS OF REVOLUTION COMPOSED OF FIBER GALSS REINFORCED PLASTIC FACINGS AND HONEYCOMB CORES, University of Oklahoma Research Institute; USAAVLABS Technical Report 67-65, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, November 1967, AD 666480.

10. Nordby, G.M., and W.C. Crisman, ANATOMY OF AN AUTOCLAVE, Reinforced Plastics and Composites World, Vol. 7, No. 2, March-April 1968, pp. 20-21.

11. Bert, C.W., W.C. Crisman, and G.M. Nordby, FABRICATION AND FULL-SCALE STRUCTURAL EVALUATION OF GLASS-FABRIC REINFORCED-PLASTIC SHELLS, Journal of Aircraft, Vol. 5, No. 1, January-February 1968, pp. 27-34.

12. Bert, C.W., W.C. Crisman, and G.M. Nordby, BUCKLING OF CYLINDRICAL AND CONICAL SANDWICH SHELLS WITH ORTHOTROPIC FACINGS, American Institute of Aeronautics and Astronautics (AIAA)/American Society of Mechanical Engineers Ninth Structures, Structural Dynamics and Materials Conference, AIAA Paper No. 68-294, Palm Springs, California, April 1968; to appear in AIAA Journal.

13. Bert, C.W., B.L. Mayberry, and J.D. Ray, BEHAVIOR OF FIBER-REINFORCED PLASTIC LAMINATES UNDER UNIAXIAL, BIAXIAL, AND SHEAR LOADINGS, University of Oklahoma Research Institute; USAAVLABS Technical Report 68-86, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia.

14. Roust, N.L., VIBRATION AND FATIGUE SANDWICH BIBLIOGRAPHY, Hexcel Corporation, Dublin, California, January 20, 1968.

15. Watkins, J.D. and R.R. Clary, VIBRATIONAL CHARACTERISTICS OF SOME THIN-WALLED CYLINDRICAL AND CONICAL FRUSTUM SHELLS, Langley Research Center; Technical Note D-2729, National Aeronautics and Space Administration, Washington, D.C., April 1965.

16. Watkins, J.D., and R.R. Clary, VIBRATIONAL CHARACTERISTICS OF THIN-WALLED CONICAL FRUSTUM SHELLS, American Institute of Aeronautics and Astronautics Journal, Vol. 2, No. 10, October 1964, pp. 1815-1816.

17. Mixson, J.S., MODES OF VIBRATION OF CONICAL FRUSTUM SHELLS WITH FREE ENDS, Journal of Spacecraft and Rockets, Vol. 4, No. 3, March 1967, pp. 414-416.

18. Hu, W.C.L., J.F. Gormley, and U.S. Lindholm, FLEXURAL VIBRATIONS OF CONICAL SHELLS WITH FREE EDGES, Southwest Research Institute; Contractor's Report CR-384, National Aeronautics and Space Administration, Washington, D.C., March 1966.

19. Dove, R.C., and P.H. Adams, EXPERIMENTAL STRESS ANALYSIS AND MOTION MEASUREMENT, Columbus, Ohio, C.E. Merrill Books, Inc., 1965, p. 226.

20. Perry, C.C., and H.R. Lissner, THE STRAIN GAGE PRIMER, New York, McGraw-Hill Book Company, Inc., 1955, p. 198.

21. Rhodes, J.E., PIEZOELECTRIC TRANSDUCER CALIBRATION SIMULATION METHOD USING SERIES VOLTAGE INSERTION, Environmental Quarterly, Vol. 8, No. 1, 1962.

22. Yu, Y.-Y., VIBRATIONS OF ELASTIC SANDWICH CYLINDRICAL SHELLS, Journal of Applied Mechanics, Vol. 27, Transactions of the American Society of Mechanical Engineers, Vol. 82E, No. 4, December 1960, pp. 653-662.

23. Chu, H.N., VIBRATIONS OF HONEYCOMB SANDWICH CYLINDERS, Journal of the Aerospace Sciences, Vol. 28, No. 12, December 1961, pp. 930-939, 944.

24. Chu, H.N., INFLUENCE OF LARGE AMPLITUDES ON FLEXURAL VIBRATIONS OF A THIN CYLINDRICAL, SANDWICH SHELL, Journal of the Aerospace Sciences, Vol. 29, No. 3, March 1962, p. 376.

25. Bieniek, M.P., and A.M. Freudenthal, FORCED VIBRATIONS OF CYLINDRICAL SANDWICH SHELLS, Journal of the Aerospace Sciences, Vol. 29, No. 2, February 1962, pp. 180-184.

26. Yu, Y.-Y., VISCOELASTIC DAMPING OF VIBRATIONS OF SANDWICH PLATES AND SHELLS, Non-Classical Shell Problems (Proceedings of the International Association for Shell Structures Symposium, Warsaw, Poland, September 2-5, 1963), ed. by W. Olszak and A. Sawczuk, Amsterdam, Holland, North-Holland Publishing Company, 1964, pp. 551-571.

27. Jones, I.W., and V.L. Salerno, THE EFFECT OF STRUCTURAL DAMPING ON THE FORCED VIBRATIONS OF CYLINDRICAL SANDWICH SHELLS, Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers, Vol. 88B, No. 3, August 1966, pp. 318-324.

28. Greenspon, J.E., EFFECT OF EXTERNAL OR INTERNAL STATIC PRESSURE ON THE NATURAL FREQUENCIES OF UNSTIFFENED, CROSS-STIFFENED, AND SANDWICH CYLINDRICAL SHELLS, Journal of the Acoustical Society of America, Vol. 39, No. 2, February 1966, pp. 407-408.

29. Mead, D.J., and A.J. Pretlove, ON THE VIBRATIONS OF CYLINDRICALLY CURVED ELASTIC SANDWICH PLATES; PART 2, THE SOLUTION FOR CYLINDRICAL PLATES, Department of Aeronautics and Astronautics, University of Southampton; Reports and Memoranda, No. 3363, Aeronautical Research Council, London, England, 1964.

30. Jacobson, M.J., and M.L. Wenner, DYNAMIC RESPONSE OF CURVED COMPOSITE PANELS IN A THERMAL ENVIRONMENT, Norair Division, Northrop Corporation; Report 66-2647, Air Force Office of Scientific Research, Washington, D.C., November 1966, AD 642941.

31. Tasi, J., EFFECT OF MASS LOSS ON THE TRANSIENT RESPONSE OF A SHALLOW SNADWICH SHELL, American Institute of Aeronautics and Astronautics Journal, Vol. 2, No. 1, January 1964, pp. 58-63.

32. Koplik, B., and Y.-Y. Yu, APPROXIMATE SOLUTIONS FOR FREQUENCIES OF AXISYMMETRIC VIBRATIONS OF SPHERICAL CAPS, Journal of Applied Mechanics, Vol. 34, Transactions of the American Society of Mechanical Engineers, Vol. 89E, September 1967, No. 3, pp. 785-787.

33. Koplik, B., and Y.-Y. Yu, AXISYMMETRIC VIBRATIONS OF HOMOGENEOUS AND SANDWICH SPHERICAL CAPS, Journal of Applied Mechanics, Vol. 34, Transactions of the American Society of Mechanical Engineers, Vol. 89E. No. 3, September 1967, pp. 667-673.

34. Koplik, B., and Y.-Y. Yu, TORSIONAL VIBRATIONS OF HOMOGENEOUS AND SANDWICH CAPS AND CIRCULAR PLATES, Department of Mechanical Engineering, Polytechnic Institute of Brooklyn; Report 67-1982, Air Force Office of Scientific Research, Washington, D.C., July 1967, AD 659491.

35. Suvernev, V.G., VIBRATION OF CIRCULAR CONICAL SANDWICH SHELLS (in Russian), Raschety elementov aviatsionnykh konstruktsiy, Trekhsloynyye paneli i obolochky, Collection of Articles, No. 4, ed. by A.Ya. Aleksandrov, E.I. Grigolyuk, and L.M. Kurshin, Moscow, USSR, Izd-vo Mashinostroyeniye, 1965, pp. 91-98.

36. Klein, V., THEORY OF PLATES AND SHELLS IN FLIGHT-VEHICLE DESIGN, Foreign Science Bulletin, Vol. 4, No. 2, February 1968, pp. 65-84, AD 66158.

37. Azar, J.J., AXISYMMETRIC FREE VIBRATIONS OF SANDWICH SHELLS OF REVOLUTION, unpublished Ph.D. dissertation, University of Oklahoma, Norman, Oklahoma, 1965.

38. Vasitsyna, T.N., FLEXURE AND FREE VIBRATION OF CYLINDRICAL SANDWICH SHELLS OF UNSYMMETRIC CONSTRUCTION (in Russian), Raschety elementov aviatsionnykh konstruktsiy, Trekhsoloynyye paneli i obolochky, Collection of Articles, No. 4, ed. by A.Ya. Aleksandrov, E.I. Grigolyuk, and L.M. Kurshin, Moscow, USSR, Izd.-vo Mashinostroyeniye, 1965.

39. Baker, E.H., and G. Herrmann, VIBRATIONS OF ORTHOTROPIC CYLINDRICAL SANDWICH SHELLS UNDER INITIAL STRESS, American Institute of Aeronautics and Astronautics Journal, Vol. 4, No. 6, June 1966, pp. 1063-1070.

40. Bacon, M.D., and C.W. Bert, UNSYMMETRIC FREE VIBRATIONS OF ORTHOTROPIC SANDWICH SHELLS OF REVOLUTION, American Institute of Aeronautics and Astronautics Journal, Vol. 5, No. 3, March 1967, pp. 413-417.

41. Hu, W.C.L., FREE VIBRATIONS OF CONICAL SHELLS, Southwest Research Institute; Technical Note D-2666, National Aeronautics and Space Administration, Washington, D.C., February 1965.

42. Hu, W.C.L., J.F. Gormley, and U.S. Lindholm, AN EXPERIMENTAL STUDY AND INEXTENSIONAL ANALYSIS OF VIBRATIONS OF FREE-FREE CONICAL SHELLS, International Journal of Mechanical Sciences, Vol. 9, 1967, pp. 123-135.

43. Krause, F.A., NATURAL FREQUENCIES AND MODE SHAPES OF THE TRUNCATED CONICAL SHELL WITH FREE EDGES, Technical Report 68-37, Air Force Space and Missile Systems Orgainization, Los Angeles Air Force Station, Los Angeles, California, January 1968, AD 665828.

44. Platus, D.H., CONICAL SHELL VIBRATIONS, Aerospace Research Associates, Inc.; Technical Note D-2767, National Aeronautics and Space Administration, Washington, D.C., April 1965.

45. Siu, C.-C., FREE VIBRATIONS OF SANDWICH CONICAL SHELLS WITH FREE EDGES, Master of Engineering (Mechanical) thesis, University of Oklahoma, Norman, Oklahoma, in progress 1968.

46. Wilkins, D.J., Jr., FREE VIBRATIONS OF ORTHOTROPIC SANDWICH CONICAL SHELLS WITH VARIOUS BOUNDARY CONDITIONS, Ph.D. dissertation, University of Oklahoma, Norman, Oklahoma, in progress 1968.

47. Watkins, J.D., and R.R. Clary, author's reply to discussion of Reference 16, American Institute of Aeronautics and Astronautics Journal, Vol. 3, No. 6, June 1965, pp. 1213-1214.

48. Hu, W.C.L., discussion of Reference 16, American Institute of Aeronautics and Astronautics Journal, Vol. 3, No. 6, June 1965, p. 1213.

49. Koval, L.R., NOTE OF EFFECT OF DYNAMIC ASYMMETRY ON THE VIBRATION OF CYLINDRICAL SHELLS, Journal of the Acoustical Society of America, Vol. 35, No. 2, February 1963, pp. 252-253.

50. Koval, L.R., NOTE ON THE VIBRATIONAL CHARACTERISTICS OF THIN-WALLED SHELLS, American Institute of Aeronautics and Astronautics Journal, Vol. 4, No. 3, March 1966, pp. 571-572.

51. Tobias, S.A., A THEORY OF IMPERFECTION FOR THE VIBRATION OF ELASTIC BODIES OF REVOLUTION, Engineering (London), Vol. 172, No. 4470, September 28, 1951, pp. 409-410.

52. Arnold, R.N., and G.B. Warburton, FLEXURAL VIBRATIONS OF THE WALLS OF THIN CYLINDRICAL SHELLS HAVING FREELY SUPPORTED EDGES, Proceedings of the Royal Society (London), Vol. 197A, 1949, pp. 238-257.

53. Mixson, J.S., EXPERIMENTAL MODES OF VIBRATION OF 14° CONICAL-FRUSTUM SHELLS, Langley Research Center; Technical Note D-4428, National Aeronautics and Space Administration, Washington, D.C., April 1968.

54. Shoua, E.D., THE COMPOSITE DAMPING CAPACITY OF SANDWICH CANTILEVER BEAMS, Experimental Mechanics, Vol. 8, No. 7, July, 1968, pp. 300-308.

55. Kennedy, C.C., and C.D.P. Pancu, USE OF VECTORS IN VIBRATION MEASUREMENT AND ANALYSIS, Journal of the Aeronautical Sciences, Vol. 14, No. 11. November, 1947, pp. 603-625.

56. Bishop, R.E.D., and G.M.L. Gladwell, AN INVESTIGATION INTO THE THEORY OF RESONANCE TESTING, Philosophical Transactions of the Royal Society (London), Vol. 255A, January 1963, pp. 241-280.

57. Pendered, J.W., and R.E.D. Bishop, A CRITICAL INTRODUCTION TO SOME INDUSTRIAL RESONANCE TESTING TECHNIQUES, Journal of Mechanical Engineering Science, Vol. 5, No. 4, 1963, pp. 345-367.

58. Pendered, J.W., and R.E.D. Bishop, THE DETERMINATION OF MODAL SHAPES IN RESONANCE TESTING, Journal of Mechanical Engineering Science, Vol. 5, No. 4, 1963, pp. 379-385.

SELECTED BIBLIOGRAPHY*

Dynamics of Isotropic Conical Shells

Agenosov, L.G., CONCERNING THE FREE VIBRATIONS OF CONICAL SHELLS (in Russian), Nonlinear Theory of Plates and Shells, Kazan State University, 1962, pp. 109-115.

Alumyae, N., REGARDING THE FUNDAMENTAL SYSTEMS OF INTEGRALS FOR THE EQUATIONS FOR SMALL AXISYMMETRIC STEADY VIBRATIONS OF AN ELASTIC CONICAL SHELL (in Russian), Izvestiia Akademii Nauk Eston. SSR, Seriya Tekh. i Fiz.-Mat. Nauk, Vol. 9, 1960, pp. 3-15.

Bagdasaryan, Zh.Ye., and V.Ts. Gnuni, RESONANCE IN FORCED NONLINEAR OSCILLATIONS OF LAMINAR ANISOTROPIC SHELLS, Izvestiia Adademii Nauk Armyanskoy SSR, Ser. Fiziko-Matematicheskikh Nauk, Vol. 41, No. 1, 1961, pp. 41-49; English translation, Technical Translation FTD-TT-61-313, Foreign Technology Division, Wright-Patterson Air Force Base, Ohio, January 23, 1962, AD 270785.

Berkowitz, H.M., and H. Bleich, AXIAL IMPACT OF AN ELASTIC RIGHT CONICAL MEMBRANE SHELL, American Institute of Aeronautics and Astronautics Journal, Vol. 4, No. 8, August 1966, pp. 1378-1384.

Bluhm, J.I., and D. Neal, ROTATIONALLY SYMMETRIC, TRANSIENT RESPONSE OF A SMALL-ANGLE TRUNCATED CONICAL SHELL OWING TO A MOVING PRESSURE FRONT, Journal of the Acoustical Society of America, Vol. 38, No. 10, October 1965, pp. 608-613.

Brady, F.H., and R.A. Newton, FREE VIBRATIONS OF CONICAL SHELLS, Report No. 9620: D0006R, Aerojet-General Corporation, Liquid Rocket Operations, Sacramento, California, August 29, 1963.

Breslavskii, V.E., SOME BASIC CASES OF FREE VIBRATIONS OF CONICAL SHELLS (in Russian), Trudy Khar' kovskogo Vysshego aviatssionnoinzhinernogo Uchilishcha, Vol. 27, 1954, pp. 107-123.

Brunkow. A.T., and R.E. Davis, CONICAL SHELL DEFLECTION AND ACCELERATION RESPONSE PROGRAM (DARC), McDonnell Aircraft Corporation; Contractor's Report CR-66013, National Aeronautics and Space Administration, Washington, D.C., July 1965.

---

*This bibliography does not include any references listed in Literature Cited. Also in the case of publications which first appeared as a thesis or report and later as a paper in a journal, only the latter form is listed, since it is the most readily accessible.

Chen, R., A RAYLEIGH-RITZ-LAGRANGE SOLUTION TO THE VIBRATION OF CONICAL SHELLS, Report No. SM-42636, Douglas Aircraft Company, Inc., Missile and Space Systems Division, Santa Monica, California, March 15, 1963.

Chen, R., COUPLED VIBRATION OF CONICAL SHELLS, Paper No. 1862, Douglas Aircraft Company, Inc., Missile and Space Systems Division, Santa Monica, California, September 1964.

Cooper, R.M., and A.F. DiGiacomo, BLAST LOAD TESTS OF CONICAL SHELLS, Technical Documentary Report No. TDR-469 (5250-30)-3, Aerospace Corporation, El Segundo, California, November 1964, AD 454615.

Coppa, A.P., SOME EXPERIMENTAL RESULTS ON THE BUCKLING BEHAVIOR OF NOSE CONE SHELL MODELS UNDER LONGITUDINAL IMPACT, Report No. R60SD358, General Electric Company, Space Sciences Laboratory, Philadelphia, Pennsylvania, 1960.

Debler, W.R., APPROXIMATE SOLUTION OF NATURAL FREQUENCIES ENCOUNTERED IN TORSIONAL VIBRATION OF A RIGHT CONICAL SHELL FRUSTUM OF CONSTANT THICKNESS WITH BUILT-IN ENDS, Unpublished Master of Science thesis, Mechanical Engineering, University of Maryland, College Park, Maryland, 1955.

Eringen, A.C., AXISYMMETRIC BUCKLING OF CONICAL SHELLS UNDER INERTIA LOADING, Technical Report No. 1-2, General Technology Corporation, Elgin, Illinois, Contract No. DA-11-022-501, April 1959, AD 216244.

Evensen, D.A., and R.E. Fulton, SOME STUDIES ON THE NONLINEAR DYNAMIC RESPONSE OF SHELL-TYPE STRUCTURES, Dynamic Stability of Structures (Proceedings of an International Conference held at Northwestern University, Evanston, Illinois, October 18-20, 1965), edited by G. Herrmann, New York, Pergamon Press, 1967, pp. 237-254.

Federhofer, K., ON FREE VIBRATIONS OF CYLINDRICAL, SPHERICAL, AND CONICAL SHELLS (in German), Proceedings of the Fifth International Congress for Applied Mechanics, Cambridge, Massachusetts, 1938, pp. 719-723.

Federhofer, K., NATURAL VIBRATIONS OF CONICAL SHELLS, Technical Translation F-8261, National Aeronautics and Space Administration, Washington, D.C., 1962.

Fulton, R.E., DYNAMIC AXISYMMETRIC BUCKLING OF SHALLOW CONICAL SHELLS SUBJECTED TO IMPULSIVE LOADS, Journal of Applied Mechanics, Vol. 22, Transactions of the American Society of Mechnical Engineers, Vol. 87E, No. 1, March 1965, pp. 129-134.

Garnet, H., M.A. Goldberg, and V.L. Salerno. TORSIONAL VIBRATIONS OF SHELLS OF REVOLUTION, Journal of Applied Mechanics, Vol. 28, Transactions of the American Society of Mechanical Engineers, Vol. 84E, No. 3, September 1962, pp. 595-596.

Gerasimov. I.S., ON A DYNAMIC SIMILARITY PROBLEM FOR A CONICAL SHELL, Soviet Physics-Doklady, Vol. 4, 1959, pp. 306-308.

Godzevish, V.G., FREE VIBRATIONS OF CIRCULAR CONIC SHELLS, Theory of Shells and Plates (Proceedings of the Fourth All-Union Conference, Erevan, October 1962), Technical Translation F-341, National Aeronautics and Space Administration, Washington, D.C., 1966, pp. 339-343.

Goldberg, J.E., AXISYMMETRIC OSCILLATION OF CONICAL SHELLS, Proceedings of the Ninth International Congress of Applied Mechanics. Brussels, Belgium, 1956, Vol. VII, pp. 333-343.

Goldberg, J.E., ON THE CALCULATION OF THE AXISYMMETRIC MODES AND FREQUENCIES OF CONICAL SHELLS, Journal of the Acoustical Society of America, Vol. 32, No. 6, June 1960, pp. 738-742.

Goldberg, J.E., and J.L. Bogdanoff, STATIC AND DYNAMIC ANALYSIS OF NON-UNIFORM CONICAL SHELLS UNDER SYMMETRICAL CONDITIONS, Proceedings of the Sixth Symposium on Ballistic Missile and Aerospace Technology, New York, Academic Press, Inc., 1961, Vol. I, pp. 219-238.

Goldberg, J.E., J.L. Bogdanoff, and D.W. Alspaugh, MODES AND FREQUENCIES OF PRESSURIZED CONICAL SHELLS, Journal of Aircraft, Vol. 1, No. 6, November-December 1964, pp. 372-374.

Grigolyuk, E.I., SMALL OSCILLATIONS OF THIN RESILIENT CONICAL SHELLS, Technical Translation F-25, National Aeronautics and Space Administration, Washington, D.C., May 1960.

Herrmann, G., and I. Mirsky, ON VIBRATIONS OF CONICAL SHELLS, Journal of the Aerospace Sciences, Vol. 25, No. 7, July 1958, pp. 451-458.

Holmes, W.J., AXISYMMETRIC VIBRATIONS OF A CONICAL SHELL SUPPORTING A MASS, Journal of the Acoustical Society of America, Vol. 34, No. 4, April 1962, pp. 458=461.

Hsu, C.S., ON DYNAMIC PROBLEMS OF SHALLOW CONICAL SHELLS, Conference on Shell Theory and Analysis, Lockheed Missiles and Space Company, Palo Alto, California, August 1963, pp. 47-49.

Hsu, C.S., and P.M. Naghdi, PROPAGATION OF THERMOELASTIC WAVES IN A SHALLOW CONICAL SHELL, Non-Classical Shell Problems (Proceedings of the International Association for Shell Structures Symposium, Warsaw, Poland, September 2-5, 1963), edited by W. Olszak and A. Sawczuk, Amsterdam, Holland, North-Holland Publishing Company, 1964, pp. 169-187.

Hu, W.C.L., and D.D. Kana, IMPEDANCE CHARACTERISTICS OF CONICAL SHELLS UNDER AXIAL EXCITATIONS, PART 1, Southwest Research Institute; Contractor's Report CR-92989, National Aeronautics and Space Administration, Washington, D.C., May 1967.

Kalnins, A., FREE VIBRATION OF ROTATIONALLY SYMMETRIC SHELLS, Journal of the Acoustical Society of America, Vol. 36, No. 7, July 1964, pp. 1355-1365.

Karnaukhov. V.G., ANALYTICAL SOLUTIONS OF PROBLEMS ON THE FREE OSCILLATIONS AND STABILITY OF CONICAL SHELLS (in Ukrainian), Prikladnia Mekhanika, Vol. 1, No. 1, 1965, pp. 12-19.

Keeffe, R.E., NATURAL FREQUENCIES OF MERIDIONAL VIBRATION IN THIN CONICAL SHELLS, American Institute of Aeronautics and Astronautics Journal, Vol. 2, No. 10, October 1964, pp. 1825-1827.

Kil'chevskiy, M.O., and A.Kh. Konstantinov, OSCILLATION OF A THICK CONICAL PANEL IN THE NON-CLASSICAL FORMULATION (in Ukrainian), Dopovidi Akademiya Nauk Ukrainskoi RSR, No. 2, 1964, pp. 194-197.

King, W.W., AXIAL IMPACT OF AN ELASTIC CONICAL MEMBRANE, Journal of Applied Mechanics, Vol. 34, Transactions of the American Society of Mechanical Engineers, Vol. 89E, No. 2, pp. 496-497.

Kovalenko, A.D., EXACT ANALYTICAL SOLUTIONS FOR SHELLS OF REVOLUTION (in Ukrainian), Prykladna Mekhanika, Vol. 10, 1964, pp. 237-246.

Lindholm, U.S., and W.C.L. Hu, NON-SYMMETRIC VIBRATIONS OF TRUNCATED CONICAL SHELLS, International Journal of Mechanical Sciences, Vol. 8, No. 9, September 1966, pp. 561-579.

Lu, S.Y., and C.L. Sun, VIBRATIONS OF THIN CONICAL SHELLS SUBJECTED TO SUDDEN HEATINGS, Journal of Aircraft, Vol. 4, No. 1, January-February, 1967, pp. 11-15.

Malkina, R.L., and V.G. Godzevich, FREE VIBRATIONS OF SHELLS OF ZERO CURVATURE (in Russian), Izvestiia Vysshikh Uchebnykh Zavedenii, Seriia Aviatsionnaia Tekhnika, 1963, No. 1, pp. 48-57.

McLachlan, N.W., ON THE SYMMETRICAL MODES OF VIBRATION OF TRUNCATED CONICAL SHELLS WITH APPLICATIONS TO LOUD-SPEAKER DIAPHRAGMS, Proceedings of the Physical Society, (London), Vol. 44, 1932, pp. 408-425.

Nash, W.A., NONLINEAR PROBLEMS IN THE DYNAMICS OF THIN SHELLS, Nonlinear Partial Differential Equations, edited by W.F. Ames, New York, Academic Press, Inc., 1967, pp. 291-307.

Partridge, G.R., MODES OF VIBRATION OF A LOUDSPEAKER CONE, unpublished Ph.D. dissertation, Yale University, New Haven, Connecticut, 1950.

Pfluger, A., STRESSES, DEFORMATIONS, AND VIBRATIONS OF CONICAL-SHAPED FLIGHT SHELLS (in German), Luftfahrtforschung, Vol. 20, 1943, pp. 29-32.

Ramakrishma, B.S., TRANSIENT RESPONSE OF SPEAKERS AND VIBRATIONS OF CONICAL SHELLS, unpublished Ph.D. dissertation, Illinois Institute of Technology, 1949.

Salnikov, G.M., STABILITY AND VIBRATIONS OF A CONICAL SHELL UNDER COMBINED LOADS (in Russian), Izvestiia Vysshikh Uchebnykh Zavedenii, Seriia Aviatsionnaia Tekhnika, 1966, No. 4, pp. 57-62.

Saunders, H., E.J. Wisniewski, and P.R. Paslay, VIBRATIONS OF CONICAL SHELLS, Journal of the Acoustical Society of America, Vol. 32, No. 6, June 1960, pp. 765-772.

Seide, P., ON THE FREE VIBRATIONS OF SIMPLY SUPPORTED TRUNCATED CONICAL SHELLS, Israel Journal of Technology, Vol. 3, February, 1965, pp. 50-61.

Shkutin, L.I., ON THE CRITERION OF DYNAMIC STABILITY OF SHALLOW SHELLS OF REVOLUTION, Theory of Shells and Plates (Proceedings of the Fourth All-Union Conference, Erevan, October 1962), Technical Translation F-341, National Aeronautics and Space Administration, Washington, D.C., 1966, pp. 936-941.

Shkutin, L.I., STABILITY OF ELASTIC SHELLS OF REVOLUTION UNDER SUDDENLY APPLIED PRESSURE, Selected Papers from a Conference on Stability Problems in Structural Mechanics (1965), Translation, Lockheed Missiles and Space Company, Sunnyvale, California, 1966, pp. 236-242, AD 626750.

Shulman, Y., VIBRATION AND FLUTTER OF CYLINDRICAL AND CONICAL SHELLS, Technical Report 74-2, Aeroelastic and Structures Research Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts, June 1959, AD 226444.

Strutt, M.J.O., ON THE AMPLITUDE OF DRIVEN LOUD SPEAKER CONES, Proceedings of the Institute of Radio Engineers, Vol. 19, No. 5, May 1931, pp. 839-850.

Strutt, M.J.O., NATURAL OSCILLATIONS OF A CONICAL SHELL (in German), Annalen der Physik, Ser. 5, Vol. 17, No. 7, 1933, pp. 729-735.

Tang, Chao-Tsien, THE VIBRATION MODES AND EIGENFREQUENCIES OF CIRCULAR CONICAL AND CYLINDRICAL SHELLS, Scientia Sinica, Vol. 13, No. 8, 1964, pp. 1189-1209.

Wang, J.T.-S., AXISYMMETRIC RESPONSE OF CONICAL SHELLS TO BLAST LOAD, Georgia Institute of Technology; Contractor's Report CR-58733, National Aeronautics and Space Administration, Washington, D.C., August 1964.

Weingarten, V.I., FREE VIBRATIONS OF CONICAL SHELLS, Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 91, No. EM 4, August 1965, pp. 69-87.

Weingarten, V.I., THE EFFECT OF INTERNAL OR EXTERNAL PRESSURE ON THE FREE VIBRATIONS OF CONICAL SHELLS, International Journal of Mechanical Sciences, Vol. 8, February 1966, pp. 115-124.

Dynamics of Orthotropic, Layered, or Stiffened Conical Shells

Burmistrov, Ye.F., NONLINEAR TRANSVERSE VIBRATIONS OF ORTHOTROPIC SHELLS OF REVOLUTION (in Russian), Inzhernyi Sbornik, Vol. 26, 1958, pp. 5-20.

Cohen, G.A., COMPUTER ANALYSIS OF ASYMMETRIC FREE VIBRATIONS OF RING-STIFFENED ORTHOTROPIC SHELLS OF REVOLUTION, American Institute of Aeronautics and Astronautics Journal, Vol. 3, No. 12, December 1965, pp. 2305-2312.

Godzevich, V.G., and O.V. Ivanova, FREE OSCILLATIONS OF CIRCULAR CONICAL AND CYLINDRICAL SHELLS REINFORCED BY RIGID CIRCULAR RIBS, Technical Translation F-291, National Aeronautics and Space Administration, Washington, D.C., February 1965.

Grigolyuk, E.I., and A.P. Mikhailov, FLUTTER OF A CIRCULAR CONICAL SHELL MADE UP OF THREE LAYERS, Soviet Physics-Doklady, Vol. 10, February 1966, pp. 793-795.

Kutnikova, V.P., and I.Ye. Sakharov, NATURAL FREQUENCIES OF VIBRATION OF ORTHOTROPIC SHALLOW CONICAL SHELLS (in Russian), Izvestiia Akademii Nauk SSSR, Otdelenie Tekhnicheskikh Nauk, Mekhanika i Mashinostroenie, No. 4, July-August 1961, pp. 183-186.

McGrattan, R.J., and E.L. North, SHELL MODE COUPLING, Report No. U411-63-005, General Dynamics Corporation, Electric Boat Division, Groton, Connecticut, March 1963, AD 409454.

Newton, R.A., FREE VIBRATIONS OF ROCKET NOZZLES, American Institute of Aeronautics and Astronautics Journal, Vol. 4, No. 7, July 1966, pp. 1303-1305.

Weingarten, V.I., FREE VIBRATIONS OF RING-STIFFENED CONICAL SHELLS, American Institute of Aeronautics and Astronautics Journal, Vol. 3, No. 8, August 1965, pp. 1475-1481.

APPENDIX

MODIFIED KENNEDY-PANCU TECHNIQUE

Kennedy and Pancu (Reference 55) introduced their method to obtain the uncoupled resonant frequencies and associated modal shapes in 1947 in connection with ground vibration tests on aircraft. Their proposals were made entirely on the basis of theoretical arguments, with experimental results mentioned but not reported in detail. Since the appearance of their paper, the technique has been used, but only a limited number of papers on the technique have appeared. All the presentations on the technique were oriented toward beam-type structures, none to the authors' knowledge dealing with shell-type structures. The technique was used with success in airplane ground vibration tests (Reference 55), with airplane inflight flutter tests (Reference 56), and with beams (Reference 57). The present application is the first known use for shell structures. The technique is especially pertinent to shell structures, etc., because of the closely coupled resonant frequencies that are often encountered in shells.

The main limitation in using the technique in vibration testing is the accuracy with which the phase between the excitation force and the output force is measured. When Kennedy and Pancu published their paper, there was no electronic equipment in existence that could measure this phase relation to the necessary accuracy. Even with accurate phase-measuring devices, there are a number of limitations in using the Kennedy-Pancu technique.

As pointed out by Pendered and Bishop (Reference 58) and also by Bishop and Gladwell (Reference 56), near resonance there is a rapid change in phase with respect to change in frequency, making the measurement of this parameter difficult. Since this is a very important parameter in determining the resonant point, its measurement must be accurate. In the present work, this difficulty was overcome by developing a testing procedure in which the phase variation was used as the control parameter. A highly accurate digital phase meter was used to measure the phase.

Distortion in the output signals causes errors in the off-resonant frequencies. When the signal-to-noise ratio becomes small, the phase signal becomes distorted with undesired signals. Fortunately, this does not present too much difficulty, since the main area of interest is near resonance where the signal-to-noise ratio is the largest. This should be enough to determine the characteristic circular arc to deduce the resonant frequency and the damping. The Lissajous pattern was used to check the quadrant of the phase enabling the determination of the circular arc extremities.

The Kennedy-Pancu analysis was derived for a system that was linear. In nearly all practical systems, some nonlinearities are inherent. This was minimized in the present tests by establishing an output level and varying the input force amplitude. In using the Kennedy-Pancu method, the Argand plots (amplitude versus phase) are normalized by the input force level; i.e.,

complex receptance is plotted on the complex plane. Thus, resonance occurs when the output force is the largest or the input force is the smallest. Holding the output level constant established the same displacement on the shell throughout the frequency range, thus minimizing the nonlinearities, regardless of whether they are geometric (large deflections) or material (nonlinear stress-strain and nonlinear damping).

The Kennedy-Pancu technique has an advantage over other methods for determining resonant frequencies because it is able to separate modes that are closer to each other and also able to come closer to determining the pure modal shape (i.e., without coupling with other modes).

In applying their technique to a multi-degree-of-freedom system (as applicable to this case), Kennedy and Pancu made the assumption that each mode acts independently of the others; i.e., each mode responds to the applied force like a single-degree-of-freedom system. Also, they assumed that the system is linear and that superposition of modes will hold. In other words, they assumed the existence of normal modes, which is a widely accepted assumption in multi-degree-of-freedom vibrational analysis.

The response of a damped single-degree-of-freedom system, with Kimball-Lovell-type material damping and excited by a simple harmonic force of amplitude F and frequency ω, can be expressed as follows

$$X = [(k - m\omega^2)^2 + b^2]^{-\frac{1}{2}} \tag{14}$$

and

$$\tan \phi = b/(k - m\omega^2) \tag{15}$$

where

b = material damping factor*

k = spring rate

m = mass

$X = X_0/F$

x = displacement, given by $X_0 \sin(\omega t - \phi)$

ϕ = phase angle between the displacement and forcing function

If a complex-plane plot is made of the complex receptance, $Xe^{i\phi}$, with the mass, m, damping factor, b, and spring rate, k, fixed and the frequency varied, the result is as shown in Figure 27.

*The material damping factor is not to be confused with the viscous damping coefficient, which is equal to b/ω.

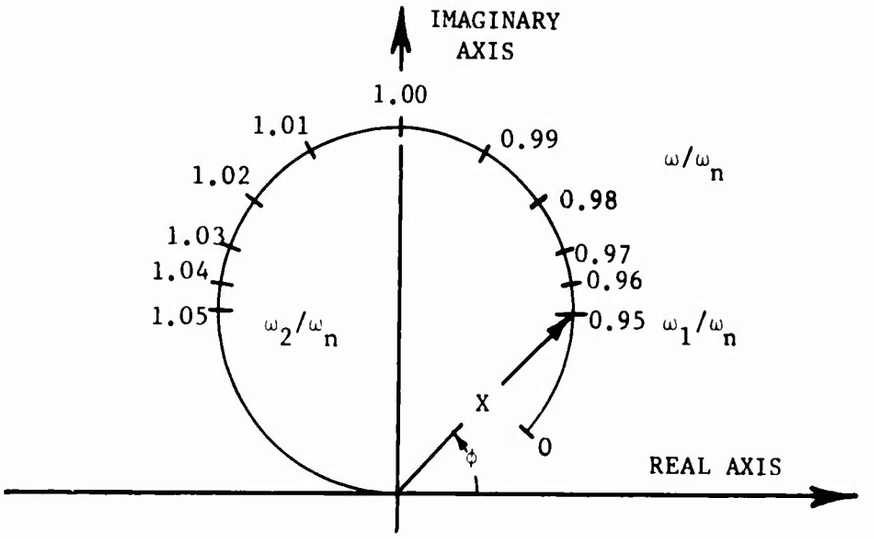


Figure 27. Argand Plot for Single-Degree-of-Freedom Damped System.

The point where the curve crosses the imaginary axis is the point in which

$$\omega^2 = \omega_n^2 \equiv k/m$$

and is defined as the resonant frequency of the system. Then at the point of resonance, the distance from the origin to the intercept of the imaginary axis is related to the material damping factor, b, as follows:

$$X_o = F/b$$

Kennedy and Pancu noted that the Argand plot traced a circular arc near the resonant point and that a circle could be passed through the points. By drawing the circle, the resonant frequency of the system would occur at the point in which the phase angle made 90 degrees with the forcing function and also the line would pass through the center of the circle. It was also noted that the resonant point occurred when the rate of change in frequency with respect to phase angle was the greatest.

The procedure for obtaining the damping coefficient from the Argand plot can be simplified by eliminating the necessity of making absolute calibrations of the excitation and response transducers. This is accomplished by expressing the damping coefficient, b, in terms of frequency as follows

$$b/k = (\omega_1 - \omega_2)/\omega_n \qquad (16)$$

where b/k is a dimensionless damping factor and ω_1 and ω_2 are the frequencies at the half-power points on the response curve. Further assuming that the

response curve is symmetrical about the resonant frequency, ω_n, then Equation (16) becomes

$$b = 2[1 - (\omega_1/\omega_n)] \tag{17}$$

The half-power-point frequency can readily be determined from the Argand plots by drawing a line from the center of the circle along a 45-degree line; the point of intersection of this line with the circle is the half-power-point frequency, ω_1. Applying these values to Equation (17), the results will give the uncoupled damping coefficient for the system.

When more than one resonance is present in the vibrating system, then each will introduce an influence on the response data. The closer the resonant frequencies are together, the greater will be the influence.

Using the assumption that each mode acts independently, then the system can be defined by a series of Equations (14) and (15), and the Argand plot of the system will be a series of circles, each circle crossing its own imaginery axis at its resonant frequency. The plot will be askew depending on how close together the resonances are, but each resonance will act independently and will have its own circle on a displaced set of coordinate axes. By applying the same reduction to each circle, the system can be separated into individual single-degree-of-freedom systems acting independently. It should be pointed out that for a continuous system the Kennedy-Pancu technique is applicable because, even though the system is continuous, the resonant frequencies are discrete.

Once the Argand plot is made, circular arcs are drawn through the data points that indicate a relatively large change in frequency with relation to change in phase. By drawing a line that is displaced 90 degrees from the forcing function and passes through the center of the circle, the intercept of this line with the circle determines the resonant frequency and the diameter (X_o) of the circle are functions of the damping factor, as mentioned before. Each circle on the plot indicates the resonant frequency and damping of the individual mode acting alone can be obtained from the distance from the center to the edge of the circle for the individual frequency points.

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13. ABSTRACT

This report describes an experimental and analytical evaluation of the vibrational characteristics of a truncated conical shell of sandwich construction. The shell was composed of fiber glass-epoxy facings and aluminum honeycomb core. The nominal shell geometry had a 5-degree semi-vertex angle, a diameter of 58 inches at the large end, and a length of 72 inches. These dimensions were selected to be typical of an Army aircraft fuselage structure. The shell was suspended by soft cords, to closely approximate free-free boundary conditions, and was excited laterally by an electrodynamic shaker placed at various locations along the shell. The shell surface was instrumented with 600 metallic-foil strain gages. Accurate separation of vibrational modes having closely spaced frequencies was accomplished by means of the Kennedy-Pancu method of data reduction, using measured frequency, strain amplitude, and relative phase angle between response and excitation. The two lowest unsymmetric-mode frequencies, for various values of circumferential wave number, agreed quite closely with those predicted by the Rayleigh-Ritz inextensional analysis presented in the report. This analysis includes the orthotropic nature of the composite-material facings. Values of the damping logarithmic decrement are also presented.

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